THEORY OF ELECTRICITY AND MAGNETISM

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HEORY OF ELECTRICITY AND MAGNETISM

Being Volume III of
"INTRODUCTION TO THEORETICAL PHYSICS"

BY

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PREFACE TO THE FIRST GERMAN EDITION

THE present book, which constitutes the third volume of my introduction to theoretical physics, discusses electric and magnetic phenomena. It is obvious that in this case, too, it was quite impossible, in view of the enormous amount of empirical material involved, which is still steadily on the increase, to achieve any degree of completeness in the treatment. It was the more important therefore to stress in our general survey the logical sequence of those lines of thought on which the system of electromagnetic theorems is founded, in order that the reader might find no difficulty in fitting into the scheme of the different kinds of phenomenon here described any case which is not actually treated, and that he might be enabled to gain access to any special literature that he might require to consult.

But the necessary uniformity and completeness of such an account can be achieved, it seems, only by using a predominatingly deductive form of treatment. I have therefore chosen two methods here, precisely as in the case of mechanics, although at the same time I have endeavoured to describe the concepts and theorems, that are introduced, as graphically as possible by discussing special examples derived from daily experience. It is in harmony with the method followed in mechanics that here, in introducing electrodynamics, we start out everywhere by assuming matter uniformly distributed in space and hence use as our basis the so-called classical theory which was founded by Maxwell and perfected by Hertz; we do not, of course, omit to refer to the characteristic limits within which this theory is valid.

Since among all the laws of physics none is so universal and so easy to grasp as the Principle of Conservation of

Energy, I have again placed it in the forefront of discussion; this leads to the further advantage that we are led to introduce quite naturally the different electric and magnetic systems of measurement, all of which are based on the energy principle. In the interests of clearness of exposition we have also emphasized the formal analysis between the electric and the magnetic vector, although it is rather external in character and results essentially, like the analogy between translation and rotation, from the circumstance, which is in a certain sense accidental, that our space happens to have three dimensions. But just as it is a fact that this analogy played an outstanding part in the historical development of Maxwell's theory, so we cannot deny that even at the present time it is still very convenient for introducing the theory and at any rate is a useful aid to memory. This is also the reason why I have used throughout the Gaussian system of measurement, which is distinguished from the rational systems of measurement used in the literature of the theory by being more closely related to the practical system of measurement. A table comparing the numerical values for several quantities in the different systems of measurement is given at the end of the book, and also an index to all the definitions and the most important theorems.

It was found possible to abbreviate the account considerably by referring to well-known theorems in mechanics, which have been derived in one of the two preceding volumes of the present series. Thus I, (132) denotes equation (132) of the first volume (General Mechanics); II, § 15 refers to section 15 of the second volume (Mechanics of Deformable Bodies). Any reader who is to any extent conversant with the principles of mechanics will not, of course, need to make use of these references.

M. PLANCK.

Berlin, Grunewald,
March, 1922.

PREFACE TO THE SECOND EDITION

The arrangement of the book, as well as the selection and distribution of the material treated, has remained unchanged in this new edition. I am fully conscious, however, of the one-sided character of this account of the subject, since in it I have by no means everywhere used the notation and formulation which are simplest and most convenient. But I do not believe that this can be regarded as a serious reproach. For in view of the manifold nature of the phenomena which are treated in the theory of electricity and magnetism, the practical requirements in the various fields are quite different, as is already shown by the circumstance that in electrostatics we find it better to use a different system of measurement from that used in electrodynamics. But, in my opinion, in giving a systematic introduction to the subject the question of convenience of notation must give way to the demand that all the concepts and theorems that are established must be built up on a single uniform foundation. Once this relationship has been clearly recognized it will require comparatively little effort later, in working out a special problem, to introduce the simplifications that are adapted to the particular case, such as, for example, the omission of the factor 4π or of the factor c, or the exchanging of the lines of induction for the lines of force, and so forth. For there is no terminology which is most convenient for *all* cases; this is a circumstance which we are bound to accept, as it lies in the nature of the subject-matter. On the other hand, the recognition of this circumstance enables us to work out the concepts far more thoroughly than can be done

if we start out from individual empirical facts, and, in particular, it enables us to make a sharp division between what is a matter of definition and what is a matter of experience. The more clearly the arbitrary factor which is contained in a definition receives expression the deeper its meaning will become rooted in the understanding.

If we adopt this as our view, there can be no doubt that there is only one fixed and certain point of departure for our exposition—namely, the concept and principle of energy. For all electric and magnetic units of measurement emerge in the last instance from the concept of energy, and all the laws which govern this region of phenomena can be deduced without difficulty from this principle. This is the reason why I placed the concepts of energy-density and energy-flux at the head of my account. They not only enable us to make an exhaustive survey of the different systems of measurement, but also enable us to deduce Maxwell's field-equations conveniently. All other consequences follow by specializing appropriately.

In this revised edition I have naturally accepted with grateful thanks many suggestions and criticisms.

M. PLANCK.

Wimpfen a.N.,
April, 1928.

TRANSLATOR'S NOTE

I wish to take this opportunity of thanking my colleagues and friends, Dr. L. G. H. Huxley, M.A., Mr. E. H. Saayman, M.A., B.Sc., as well as Mr. J. E. Keyston, B.Sc., of the Electrical Laboratory, Oxford, for helpful suggestions and for reading the proofs. In the index certain German expressions have been included for the sake of those who wish to refer to German sources.

HENRY L. BROSE.

Nottingham,
April, 1932.

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INTRODUCTION

§ 1. In addition to the mechanical processes or motions of material points there are the electrical and magnetic or electro-dynamic phenomena which form an equally uniform whole, but are definitely distinct from them. Moreover, these two regions completely cover the whole field of physics, for all the other parts, acoustics, optics and heat, can be traced back to mechanics and electrodynamics. On the other hand, the complete fusion of the last two groups of phenomena, which would appear to be the ultimate goal of physics, is as yet reserved for the future.

Nevertheless, there are even now a number of bridges which lead from the one region to the other. The first and most important of these is the *Principle of the Conservation of Energy* (I, § 49), to which we shall therefore give priority of position. This principle taken alone does not, of course, give us a sufficient hold to link up with it a definite theory of electricity. Rather, in the course of time several theories, all of which rest on the energy principle, have competed with each other. The characteristic feature of the theory which is presented in this book and which was founded by Maxwell is given by a second fundamental idea: that of the *Principle of Contiguous Action* (Action by Contact). According to this principle there is no such thing in nature as a causal action at a distance or across space; that is, it is impossible for the action of a local event to make itself felt at a more or less distant place by jumping across the intervening bodies. Rather, every causal action propagates itself from point

to point with a finite velocity through space. From this it follows that everything which happens at a certain place at a certain time is completely and uniquely defined by the events which have occurred immediately preceding it in the immediate neighbourhood of the place.

Since this proposition essentially restricts the possible ways in which physical causes can take effect, the principles of action at a distance and of contiguous action ("far" action and "near" action) are by no means to be regarded as co-ordinated; rather, the principle of action at a distance is of a more general nature, whereas that of a contiguous action is rather more special. It is owing to this that there have been several different theories of action at a distance in electrodynamics, but only one of contiguous action—namely, that of Maxwell. This theory, then, owes its sovereign position over all others not to its greater "correctness," but rather to its greater definiteness and simplicity. For it states that when we wish to make calculations about events at a certain place we need not, in principle, take account of what occurs at another place which is at a finite distance away, but need only restrict ourselves to a consideration of the events that occur in the immediate vicinity. If we admit the principle of action at a distance, on the other hand, we are, strictly speaking, compelled to search throughout the whole universe to see whether the events to be calculated are not influenced directly to an appreciable extent by an event at some other place.

This, again, confirms the statement that the more special a theory is, and *not* the more general, the more definite are the answers which it gives to all fundamental questions and the more it will fulfil its proper task, which surely consists in making an *unambiguous* assertion about the phenomena that are expected to occur. This is a point which is unfortunately often overlooked in theoretical speculations. The fewer the undetermined constants that a theory contains the more it achieves.

If in the following developments we are then com-

pelled by our principle to renounce every idea of a direct action at a distance, this need not hinder us on later occasions from finding it useful and convenient to state a result that has been finally arrived at in a form which is borrowed from the idea of a direct action at a distance. But this does not mean that we are contradicting the Principle of Contiguous Action. For example, we speak of the rising and the setting sun, but we do not by so doing imply that the sun revolves around the earth. In the same way we shall in the sequel permit ourselves to speak of an electrically charged body attracting or repelling another charged body, or a galvanic current deflecting a compass needle, on the tacit understanding that this easily pictured action represents the final result of a great number of complicated processes which take place in the space between the bodies in question.

PART ONE

GENERAL EQUATIONS OF THE ELECTROMAGNETIC FIELD IN STATIONARY BODIES

CHAPTER I

ELECTRIC AND MAGNETIC INTENSITY OF FIELD

§ 2. If a rod of sealing-wax or ebonite is vigorously rubbed with a cat's skin, the rod acquires the property of attracting to itself small and light bodies such as paper-clippings or pith-balls. To exploit this experimental fact scientifically, we do not imagine the rubbed rod to exert an attractive force on the small bodies, but start out from the point of view of the theory of contiguous action and say that the rubbed rod first produces in the air surrounding it an *electrical field* which is characterized at each of its points by certain local properties, and this electric field exerts on every body situated in it a force which depends only on the properties of the field at the particular point where the body happens to be situated. With this view we need no longer trouble about how this field has come about or how it behaves at other points. Electric fields can be produced in very different ways in all substances, including liquids and solids; for this reason every substance is to be regarded as a "dielectric." At the same time, electric fields in different substances often exhibit very different properties, particularly with regard to variation in time.

The main question is now this: What quantities characterize an electric field at a point when it has been produced in some way by a body? To answer this question we take a small, appropriately prepared "test-body," say a pith-ball hanging by a thin silk thread which has previously been brought into contact with a rubbed rod of ebonite or with the cat's skin used in the rubbing,

We then bring this ball to the point of the field in question and measure the mechanical force exerted on it. When we have in this way tested the whole field by carrying out this measurement at all points, we regard the field as fully known and defined. If the force at all points of the field is the same, the field is called "homogeneous." But in general it will vary from point to point in magnitude and direction. Since force is represented by a vector, we also characterize the electric field at each of its points by means of a vector, which we call the electric "intensity of field" $\boldsymbol{E}$. But we must bear in mind that the mechanical force that is measured depends not only on the nature of the field, but also on the nature of the test-body that is used—both as regards magnitude and direction. For the force is the greater the more strongly the ebonite rod which has touched the pith-ball has been rubbed; and, also, the force acts in two exactly opposite directions, according as the test-body has been brought into contact with the rod or the cat's skin.

It has been agreed to regard as the direction of $\boldsymbol{E}$ that in which the force acts when the test-body has been in contact with the cat's skin. (It may be remarked incidentally that the opposite convention would have been more convenient.)

It is not so easy to define the absolute value (E) of the electric intensity of field. To arrive at it we start out from the concept of the energy of the field. The conclusion that an electric field contains a certain supply of energy emerges from the fact that the field can set a body into motion. For according to the universal principle of the conservation of energy, the *vis viva* of the motion can be supplied only by the electrical energy of the field; just as the motions of an elastic body are furnished by the energy of deformation. The ratio of the electrical energy contained in an infinitely small part of the field to the volume of this part of the field is called the "electrical energy-density" of the field. This is a quantity which can be expressed in mechanical measure and is positive

if we set the energy of the electrically neutral field equal to zero. We link up with it the definition of the absolute value of the electric intensity of field by setting the electrical energy density proportional to the square of the electrical intensity of field E^2 and to a positive factor of proportionality ϵ, the "dielectric constant", which we shall define later (§ 7), only remarking here that it depends on the material constitution of the medium (for example, air).

It would be simplest to set the energy-density equal to $\frac{\epsilon}{2} \cdot E^2$. This is done in the so-called rational systems of measurement which are particularly used in theoretical physics. Owing to the historical course of development of the theory of electricity, however, it has come about that in the units which have been introduced in practice the electrical density of energy is given the value:

$$\frac{\epsilon}{8\pi} \cdot E^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

and since experimental and technical physics is still founded on these units, we shall also use them here. The particular advantages they offer will, however, only appear later (end of § 41).

The magnitude and direction of the electric intensity E also determine the components E_x, E_y, E_z of this vector with respect to a rectangular right-handed co-ordinate system. Hence the total energy of any arbitrary electric field in a homogeneous body of dielectric constant ϵ is:

$$\frac{\epsilon}{8\pi} \int d\tau (E_x{}^2 + E_y{}^2 + E_z{}^2) \quad . \quad . \quad . \quad . \qquad (2)$$

where $d\tau$ denotes the element of volume of the body.

§ 3. The *magnetic field* is analogous to the electric field, although different in its nature. Magnetic fields may also be produced in many different ways. But we shall not for the present enquire into these ways, but rather into the characteristic properties of any given magnetic field,

and we shall use as a test-body to probe the field a small magnetic needle which can be freely rotated about its centre of gravity. The mechanical turning moment exerted by the field on the magnetic needle then gives us a means of determining the "magnetic field-strength" $\boldsymbol{H}$. If this turning moment is everywhere the same, the magnetic field is called "homogeneous." In the general case we define the direction of the vector as that into which the testing needle adjusts itself, the direction being taken from the south to the north pole. Hence, for example, the direction of the magnetic field at the surface of the earth almost coincides with the direction of the geographical north.

The absolute value of the magnetic intensity of field $\boldsymbol{H}$ cannot be determined by the magnitude of the mechanical turning moment exerted on the testing needle, since this depends on the nature of the needle. We therefore define it in terms of the energy of the field by setting this energy equal to:

$$\frac{\mu}{8\pi}\cdot \boldsymbol{H}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where μ, the "magnetic permeability" of the substance, stands for a factor of proportionality which is to be defined later.

The magnitude and the direction of the magnetic intensity of field also fix the components $\boldsymbol{H}_x$, $\boldsymbol{H}_y$, $\boldsymbol{H}_z$ of the vector with respect to a rectangular right-handed co-ordinate system. The total energy of any magnetic field in a body of constant permeability μ is accordingly:

$$\frac{\mu}{8\pi}\int d\tau(\boldsymbol{H}_x^2 + \boldsymbol{H}_y^2 + \boldsymbol{H}_z^2) \quad . \quad . \quad . \quad (4)$$

In general, a field exerts both electrical *and* magnetic action simultaneously. We therefore call the field *electromagnetic*. The electromagnetic state at any point of the field is characterized by the two vectors $\boldsymbol{E}$ and $\boldsymbol{H}$, which are for the present completely independent of each

other. The total energy of the electromagnetic field is also determined by them—namely, by the sum of the expressions (2) and (4). In this sense the whole universe forms a single electromagnetic field, and all the electric and magnetic processes are nothing more than changes in this field. The general object of the theory then ultimately resolves itself into calculating the time changes of the electric and magnetic intensities of the field at all points of space, when they are given for some particular moment of time. Since the two vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ contain six quantities which are independent of one another, six equations are required to calculate their changes in time. We shall establish these six differential equations, which form the nucleus of Maxwell's theory, in the next chapter.

CHAPTER II

LAWS OF THE ELECTROMAGNETIC FIELD

§ 4. IT is obvious that the differential equations of the electromagnetic field cannot be obtained by purely deductive methods. But they emerge in a comparatively simple form if we start out from a postulate which these equations must satisfy from the very outset. This postulate is that they must satisfy, first, the principle of the conservation of energy; secondly, the principle of contiguous action. Let us consider any arbitrary portion of a homogeneous body which is permanently at rest and in which there is an electromagnetic field. According to the principle of conservation of energy, the electromagnetic energy of this part of the body can become changed only if there is an exchange of energy with outside bodies or a transformation of the energy in the interior into other forms of energy. We shall first consider the former process—that is, the reception of energy *from* the surroundings or the transference of energy *to* the surroundings. According to the principle of contiguous action, the electromagnetic energy can under no circumstances pass abruptly from any place in the surroundings to any place in the interior of the field under consideration; rather it can enter only by flowing continuously from the outside through the surface into the interior of the space in question. Thus the exchange of energy with the surroundings is regulated by a flux of energy, analogous to the flow of a fluid, through the surface of the enclosed space, and this flow of electromagnetic energy is completely determined at every point by the electromagnetic state at that point—that is, by the values of $\boldsymbol{E}$ and $\boldsymbol{H}$ at

the point in question. The amount of energy which flows during the element of time dt through the element of surface $d\sigma$ in the direction of the normal ν will be proportional to $d\sigma$ and dt, so that we set it equal to :

$$\mathbf{S}_\nu \,.\, d\sigma \,.\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

and we call the finite quantity $\mathbf{S}_\nu$ the component of the energy flux in the direction ν. It can then easily be proved that $\mathbf{S}$ is a vector. For if we apply the principle of the conservation of energy to an infinitely small tetrahedron in which three of the faces are taken respectively parallel to the three co-ordinate planes, so that the inner normals coincide with the positive directions of the co-ordinates x, y, z, whereas the fourth face has for its inner normal the direction ν (exactly as in II, § 17), then the total amount of energy which flows from without during the time dt through the surface into the tetrahedron is, by (5) :

$$(\mathbf{S}_x d\sigma_x + \mathbf{S}_y d\sigma_y + \mathbf{S}_z d\sigma_z + \mathbf{S}_\nu d\sigma)dt \quad . \quad . \qquad (6)$$

where the areas of the faces are :

$$d\sigma_x = -\,d\sigma \,.\, \cos(\nu x),\; d\sigma_y = -\,d\sigma \,.\, \cos(\nu y) \quad . \qquad (7)$$
$$d\sigma_z = -\,d\sigma \,.\, \cos(\nu z)$$

According to the energy principle the expression (6) gives the change that occurs during the time dt in the total energy contained in the tetrahedron. But since this total energy is at any rate proportional to the volume of the tetrahedron and hence is of the third order of infinitely small quantities with respect to the space dimensions, whereas every member of the sum in (6) is of the second order of infinitely small quantities, it follows that the quantity (6) vanishes, or, in view of (7), we get :

$$\mathbf{S}_\nu = \mathbf{S}_x \cos(\nu x) + \mathbf{S}_y \cos(\nu y) + \mathbf{S}_z \cos(\nu z) \quad . \qquad (8)$$

That is, by I (40) the quantity $\mathbf{S}_\nu$ is the component in the direction of the vector $\mathbf{S}$ which is defined by the components $\mathbf{S}_x$, $\mathbf{S}_y$, $\mathbf{S}_z$ in the directions of the co-ordinate axes. By the theory of contiguous action this vector $\mathbf{S}$,

the vector of the flux of electromagnetic energy, is determined at every point of the field by the values of the two vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ at the point in question.

§ 5. The manner in which the energy flux $\boldsymbol{S}$ depends on the field-strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ must be deduced from experience. It has been shewn to be regulated by a very simple law which we take as our main pillar in building up the electromagnetic field equations, since it is the most comprehensive and exhaustive expression of all the experimental results gathered in this sphere. This law is Poynting's Law of Energy Flux, which states that $\boldsymbol{S}$ *is proportional to the vector product* (I, § 87) of $\boldsymbol{E}$ and $\boldsymbol{H}$, that is:

$$\boldsymbol{S} = \frac{C}{4\pi}.[\boldsymbol{E}, \boldsymbol{H}] \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or, what comes to the same thing:

$$S_x = \frac{C}{4\pi}(E_yH_z - E_zH_y),\ S_y = \frac{C}{4\pi}(E_zH_x - E_xH_z) \quad . \quad . \quad (10)$$

where C is a certain factor of proportionality, whose value is dependent on the choice of the units for $\boldsymbol{E}$ and $\boldsymbol{H}$.

Although these relations have the disadvantage that we cannot easily visualize them pictorially, this is more than counterbalanced by the fact that from them, as we shall see later, definite quantitative laws can be derived for all electric and magnetic processes in a simple manner without introducing any particular experimental results.

§ 6. On the basis of the conventions so far made, we are now in a position to decide finally what units are to be used in the sequel. Let us first epitomize the definitions that we have set up by fixing our attention on a given electromagnetic field in a given medium. At every point of this field we have a definite density of electrical energy (1), a definite density of magnetic energy (3) and a definite flux of energy (9). These are three quantities which, although only indirectly measurable, can be expressed quite definitely in mechanical measure. Besides the two variable field-strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ they also contain the

three constants of proportionality ϵ, μ and C, concerning whose value we have not yet made a decision. From this it follows that of these three constants two can and, in fact, *must* be arbitrarily fixed in order that the two field-strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ and the third constant may be completely defined by the three energy expressions. This holds for each medium individually, for the definition can be made for each independently of the other media.

With regard to the conditions in the different kinds of media, it has been found expedient to assume the *factor of proportionality* C *of the energy flux to have the same value in all the different media.* The advantage of this convention is seen immediately in the simple form which the boundary conditions assume at the surface of separation of two media of different kinds. For if we consider a surface-element $d\sigma$ of such a surface of separation and take the z-axis in one of the directions normal to $d\sigma$, then, by (5), the energy-principle requires that the z-component of the energy flux $\boldsymbol{S}_z$ shall have the same value on both sides of the surface of separation. For otherwise energy would accumulate in the surface-element $d\sigma$ and would pass into nothingness, or would arise out of nothingness. Thus $\boldsymbol{S}_z = \boldsymbol{S'}_z$ or, by (10), since we have $C' = C$ by our convention, we get:

$$\boldsymbol{E}_x\boldsymbol{H}_y - \boldsymbol{E}_y\boldsymbol{H}_x = \boldsymbol{E}_{x'}\boldsymbol{H}_{y'} - \boldsymbol{E}_{y'}\boldsymbol{H}_{x'}$$

where the accent distinguishes the quantities for the other medium.

But since the components of the electric and the magnetic field-strength are completely independent of each other in every medium, this equation is generally satisfied only if:

$$\boldsymbol{E}_x = \boldsymbol{E}_{x'}, \boldsymbol{E}_y = \boldsymbol{E}_{y'}, \boldsymbol{H}_x = \boldsymbol{H}_{x'}, \boldsymbol{H}_y = \boldsymbol{H}_{y'}$$

always, or, more briefly, if τ stands for any direction which is tangential to the surface of separation, if:

$$\boldsymbol{E}_\tau = \boldsymbol{E}_{\tau'}, \boldsymbol{H}_\tau = \boldsymbol{H}_{\tau'} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence we can combine the general boundary conditions

at the surface of separation of two different media in the one law : *the tangential components of the electric and the magnetic intensities are continuous.*

§ 7. According to the result just obtained, the electric and the magnetic field-strength are defined for media of every kind when they are defined for a single medium. For since the direction of the resultant field-strength is already completely fixed from the beginning, the magnitude of the normal component is known if the magnitude of the tangential component is given. Hence to complete the definitions it only remains for us to fix arbitrarily two of the constants of proportionality ϵ, μ, C for a single medium which is to be arbitrarily chosen as the body of reference. We choose as this medium of reference the so-called absolute vacuum, which is also called the "pure ether." Actually there is no absolute vacuum in nature. For even the medium which approximates to it most closely—namely, inter-stellar space—certainly contains traces of ponderable matter in all parts of it. But it is one of the most important facts of electrodynamics, and one which has received support by experiments of the most varied kind, that the electromagnetic properties of a space which is poor in matter approach a limit which can be clearly specified and which is quite independent of the constitution of the remaining matter in the space, when it is further evacuated. The medium to which these limiting properties tend is called the absolute vacuum.

We denote the dielectric constant of the vacuum by ϵ_0, its permeability by μ_0. The constant C can be left without an index, since it is the same for all media. Two of these three constants are to be fixed arbitrarily. We shall therefore set them equal to 1. This then also fixes the third constant. According to the way in which the remaining choice is made, we arrive at the three different classical electromagnetic systems of measurement.

1. In the *Gaussian system* $\epsilon_0 = 1$ and $\mu_0 = 1$. This makes C assume a certain definite value c, which, as we

shall shortly see, can be measured in various ways. The constant c is not a pure number. We find its dimensions by reflecting that, since in the Gaussian system of units ϵ_0 and μ_0 are pure numbers, the electric and the magnetic intensities of field have, by (1) and (3), the dimensions:

$$\sqrt{\text{energy-density}} = [M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}] \quad . \quad . \quad (12)$$

Now since by (5) energy-flux has the dimensions (MT^{-3}), it follows from (9) or (10) that the constant c has the dimensions $[LT^{-1}]$. Thus c has the dimensions of a velocity; that is, if we change the units of mass, length and time the numerical value of c alters in such a way that it always denotes the same velocity. This definite value of the velocity is called the critical velocity. We shall get to know its numerical value when we discuss the first method of measurement (§ 60).

2. In *Maxwell's electrostatic system of units* $\epsilon'_0 = 1$ and $C' = 1$. The electric intensity of field thus again acquires the dimensions (12), whereas the magnetic intensity of field, which must now be calculated from the energy-flux (9), acquires the dimensions:

$$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

so that the magnetic permeability μ_0' has the dimensions $[L^{-2}T^2]$ by (3); that is, its dimensions are the reciprocal of the square of a velocity.

3. In *Maxwell's electromagnetic system of units* $\mu''_0 = 1$ and $C'' = 1$. The electric and the magnetic intensities now reverse their rôles. The electric intensity of field assumes the dimensions (13), while the magnetic intensity is represented by (12). The dielectric constant ϵ_0'' is represented by the square of the reciprocal of a velocity.

To find the quantitative relationships between the numerical values of a definite physical quantity in the different systems of measurement, we imagine a definite electromagnetic field in the vacuum and write down the expressions for the values of the energy in the three

systems, which we again distinguish by means of accents. By (1) we have for the density of electrical energy :

$$\epsilon_0 E^2 = \epsilon'_0 E'^2 = \epsilon''_0 E''^2 \quad . \quad . \quad . \quad . \quad (14a)$$

for the density of magnetic energy, by (3) :

$$\mu_0 H^2 = \mu'_0 H'^2 = \mu''_0 H''^2 \quad . \quad . \quad . \quad (14b)$$

and for the energy-flux, by (9) :

$$CEH = C'E'H' = C''E''H'' \quad . \quad . \quad . \quad (14c)$$

Here we have made use of the fact that the directions of the three vectors E, H, S are the same in the three systems of measurement.

If we substitute in these six equations the values given above for the individual constants, simple calculation gives the relations :

$$\epsilon''_0 = \frac{1}{c^2},\ \mu'_0 = \frac{1}{c^2} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$E = E' = \frac{E''}{c},\ H = \frac{H'}{c} = H'' \quad . \quad . \quad . \quad (16)$$

in which the differences in the numerical values for the different systems of measurement are all reduced to terms of the critical velocity c. But the equations (16) hold not only for a vacuum but, on account of the boundary conditions (11), also for any arbitrary medium. From this it follows that if we apply the equations (14) to any arbitrary medium :

$$\epsilon = \epsilon' = c^2\epsilon'',\ \mu = c^2\mu' = \mu'' \quad . \quad . \quad . \quad (17)$$

Since all the concepts that are used in electrodynamics are to be derived from those of the intensity of field and of the constants ϵ and μ, the equations (16) and (17) give us the key to understanding the relationships between the different systems of measurement.

In addition to the three classical absolute systems of measurement, we also have the *practical system of measurement* which links up closely with Maxwell's electromagnetic system and is distinguished from it only in having its

number values multiplied for practical reasons by powers of ten. We shall revert to this point later.

In theoretical physics we often find Lorentz's *rational system of units* in use. This is characterized by the fact that, as already mentioned above, the factor 4π is missing in the denominator in the expressions (1), (3), (9) for the energy-density and the energy-flux, whereas the dielectric constant and the magnetic permeability are defined exactly as in the Gaussian system. If we denote the quantities measured in Lorentz's rational system by a horizontal bar, then :

$$\epsilon = \bar{\epsilon}, \; \mu = \bar{\mu} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and hence, if as in (14) we write down the energy-expressions in the Gaussian and the Lorentz system, we have :

$$\frac{E}{\sqrt{4\pi}} = \bar{E}, \quad \frac{H}{\sqrt{4\pi}} = \bar{H} \quad . \quad . \quad . \quad (19)$$

A table of the relations which hold between the values of the most important electrical and magnetic quantities when measured in different systems of units is given at the end of the book. To avoid confusion we shall exclusively use Gauss's system in the present volume.

The fact that when a definite physical quantity is measured in two different systems of units it has not only different numerical values, but also different dimensions has often been interpreted as an inconsistency that demands explanation, and has given rise to the question of the "real" dimensions of a physical quantity. After the above discussion it is clear that this question has no more sense than inquiring into the "real" name of an object (cf. I, § 28).

§ 8. We now discuss the second of the two causes mentioned in § 4 which may account for a change in the electromagnetic energy contained in a body—namely, its transformation into other forms of energy. In this case, too, all that is known can be compressed into a single sentence : in every medium electrical energy is constantly

being transformed into heat; the energy which in the time dt and in the element of volume $d\tau$ becomes transformed into heat is proportional to the momentary local electrical energy-density:

$$dt \, . \, d\tau \cdot \frac{\epsilon}{8\pi} \cdot E^2 \, . \, \text{const.}$$

This quantity of energy is called its "Joule heat" and the process can be pictured to some extent by its analogy with the transformation of elastic energy into heat, such as occurs in a deformed incompletely elastic body when the elastic forces of tension gradually subside.

A simple dimensional calculation shows that the constant of proportionality which occurs in the last expression and which is dependent on the nature of the medium represents the reciprocal of a time; hence we denote it by $\frac{2}{T}$. The equivalent Joule heat is then:

$$dt \, . \, d\tau \cdot \frac{\epsilon}{4\pi T} \cdot E^2 = dt \, . \, d\tau \, . \, \kappa \, . \, E^2 \qquad (20)$$

where we have used the abbreviation:

$$\frac{\epsilon}{4\pi T} = \kappa. \qquad (21)$$

The greater the value of T the longer it takes to use up the electrical energy. Hence T is also called the "time of relaxation" of the medium. For metals T is extremely small, but for gases it is very great, and for the absolute vacuum it is infinitely great; that is, electrical energy can maintain itself for an infinite time in a vacuum just as elastic stresses can persist for an infinite time in a completely elastic body.

There is no process analogous to relaxation in the case of magnetic energy.

§ 9. We are now sufficiently prepared to follow out the line of reasoning indicated in § 4 and to obtain the general equations of the electromagnetic equations by applying the energy-principle. The change which occurs during

the element of time dt in the electromagnetic energy contained in any part of a homogeneous body is, by (2) and (4):

$$\frac{dt}{4\pi} \cdot \int d\tau . \{\epsilon(E_x\dot{E}_x + E_y\dot{E}_y + E_z\dot{E}_z) + \mu(H_x\dot{H}_x + H_y\dot{H}_y + H_z\dot{H}_z)\} \quad (22)$$

This change is due in the first place to the energy (5) which flows in the same time from without through the surface into the space occupied by the body: its total value is:

$$dt . \int d\sigma . S_\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and secondly by the simultaneous generation of heat (20) which amounts to:

$$dt . \int d\tau \kappa E^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

And the expression (22) is equal to the expression (23) minus the expression (24).

If we next transform the surface integral (23) into a space integral by applying (8) in accordance with II, 78, we get:

$$\int d\sigma . S_\nu = -\int d\tau \left(\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + \frac{\partial S_z}{\partial z}\right) = -\int d\tau . \operatorname{div} S \quad (25)$$

If we then transfer all quantities to the left-hand side of the equation, and include them all in a single space-integral the quantity multiplied by $d\tau$ must vanish, since the portion of space under consideration can be taken as small as we please. This gives:

$$\frac{\epsilon}{4\pi}(E_x\dot{E}_x + E_y\dot{E}_y + E_z\dot{E}_z) + \frac{\mu}{4\pi}(H_x\dot{H}_x + H_y\dot{H}_y + H_z\dot{H}_z)$$
$$+ \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + \frac{\partial S_z}{\partial z} + \kappa(E_x^2 + E_y^2 + E_z^2) = 0 \quad . \quad (25a)$$

where the vector S of energy-flux is given by (9) in the Gaussian system:

$$S = \frac{c}{4\pi}[E, H] \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The equation (25a), which holds generally, is a homogeneous quadratic function of the six intensities of field and their derivatives. To obtain from it the six homogeneous linear differential equations of the electromagnetic field it suggests itself to us to equate to zero those of the quantities in it which are multiplied by the six field-components E_x, E_y, E_z, H_x, H_y, H_z. We then get the six equations:

$$\epsilon\dot{E}_x = c\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) - 4\pi\kappa E_x, \ldots$$

$$\mu\dot{H}_x = -c\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right), \ldots$$

or, in vectorial form, by II (65):

$$\epsilon\dot{E} = c \operatorname{curl} H - 4\pi\kappa E \quad . \quad . \quad . \quad (27a)$$

$$\mu\dot{H} = -c \operatorname{curl} E \quad . \quad . \quad . \quad . \quad . \quad (27b)$$

and these are *Maxwell's fundamental equations for the electromagnetic field* in a homogeneous isotropic body at rest. We shall see that all the laws of electric and magnetic phenomena in such bodies can be uniquely derived from Maxwell's equations and the boundary conditions (11).

If, for brevity, we introduce the vectors:

$$\epsilon E = D \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$\kappa E = J \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$\mu H = B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

the field-equations assume the form:

$$\dot{D} = c \operatorname{curl} H - 4\pi J \quad . \quad . \quad . \quad . \quad (31a)$$

$$\dot{B} = -c \operatorname{curl} E \quad . \quad . \quad . \quad . \quad . \quad (31b)$$

which are distinguished by the fact that they contain no constant which refers to the particular nature of the body, and so have a universal character. Hence this form of the equations can be suitably applied also to *non-homogeneous and non-isotropic bodies* so long as they are continuously extended and may be assumed to be at rest. Within the range thus specified these equations have been

found to hold extremely well wherever they have been applied. They form, so to speak, the solid framework into which all the individual peculiarities conditioned by the special properties of various bodies can be fitted. These peculiarities find expression only in relationships such as those which are assumed for homogeneous isotropic bodies by the equations (28), (29), (30).

Thus the whole system of optics of crystals is derived by a simple modification of the relation (28), which consists in replacing the simple proportionality of the vectors $\boldsymbol{E}$ and $\boldsymbol{D}$ by a more general linear relation between the respective components (cf. IV, § 52). Further, we arrive at a theory of electrical excitation in a non-uniformly concentrated solution by generalizing the equation (29). This is done by adding to the electrical intensity of field $\boldsymbol{E}$ a quantity called the "exciting" electromotive force. Finally, the laws of ferromagnetic substances are obtained by appropriately generalizing the relation (30) (cf. § 35, below).

It is understood that each of these generalizations of the equations (28), (29) and (30) also affects the expressions (2), (4) and (20) for the energy-densities and the Joule heat, respectively, whereas on the other hand the expression (26) for the energy-flux remains unaffected by the generalization, since no material constant is contained in (26). The particular modification in each case is determined by applying the energy-principle as in § 9, with the help of the universal equations (26) and (31).

From this there follows, for example, for the change of electrical energy in the time dt:

$$\frac{1}{4\pi}\int d\tau(\boldsymbol{E}_x d\boldsymbol{D}_x + \boldsymbol{E}_y d\boldsymbol{D}_y + \boldsymbol{E}_z d\boldsymbol{D}_z) \quad . \quad . \quad (32)$$

and for the change of magnetic energy:

$$\frac{1}{4\pi}\int d\tau(\boldsymbol{H}_x d\boldsymbol{B}_x + \boldsymbol{H}_y d\boldsymbol{B}_y + \boldsymbol{H}_z d\boldsymbol{B}_z) \quad . \quad . \quad (32a)$$

which are generalizations of the expressions (2) and (4).

§ 10. Although the boundary conditions (11) contain only the tangential components of the intensities of field, in combination with the equations for the interior they allow us to draw a conclusion about the relations between the normal components of the field on both sides of the surface of separation of two media. For let us consider again, as in § 6, a point in the surface of separation between two media, choosing the tangential plane as the xy-plane. Then by (31a) the equations for the normal component of the electric vector $\boldsymbol{D}$ on the sides of the surface of separation are:

$$\frac{\partial \boldsymbol{D}_z}{\partial t} = c\left(\frac{\partial \boldsymbol{H}_y}{\partial x} - \frac{\partial \boldsymbol{H}_x}{\partial y}\right) - 4\pi \boldsymbol{J}_z$$

and:

$$\frac{\partial \boldsymbol{D}'_z}{\partial t} = c\left(\frac{\partial \boldsymbol{H}'_y}{\partial x} - \frac{\partial \boldsymbol{H}'_x}{\partial y}\right) - 4\pi \boldsymbol{J}'_z$$

so that by subtraction we have:

$$\frac{\partial}{\partial t}(\boldsymbol{D}'_z - \boldsymbol{D}_z) = c\left(\frac{\partial(\boldsymbol{H}'_y - \boldsymbol{H}_y)}{\partial x} - \frac{\partial(\boldsymbol{H}'_x - \boldsymbol{H}_x)}{\partial y}\right) - 4\pi(\boldsymbol{J}'_z - \boldsymbol{J}_z) \quad (32b)$$

According to the boundary conditions (11) the differences of the tangential magnetic components are:

$$\boldsymbol{H}'_y - \boldsymbol{H}_y = 0 \text{ and } \boldsymbol{H}'_x - \boldsymbol{H}_x = 0 \quad . \quad . \quad (33)$$

Not only these differences themselves, however, but also their derivatives with respect to x and y are equal to zero, because the equations (33) also hold for all neighbouring points of the surface of separation and hence may be differentiated with respect to any direction which, like the directions x and y, lie in the tangential plane; but differentiation with respect to the normal z is, of course, not allowed, since the equations (33) are valid only for the one value $z = 0$.

Hence in equation (32b) the term involving c drops out altogether, and we obtain in general for the normal components of $\boldsymbol{D}$ on both sides of the surface of separation:

$$\frac{\partial}{\partial t}(\boldsymbol{D}'_\nu - \boldsymbol{D}_\nu) = -4\pi(\boldsymbol{J}'_\nu - \boldsymbol{J}_\nu)$$

This equation may with advantage be written in a symmetrical form by using the normal $\nu' = -\nu$ directed towards the interior of the medium for the components in the accented system :

$$\frac{\partial}{\partial t}(\boldsymbol{D}'_{\nu'} + \boldsymbol{D}_\nu) = -4\pi(\boldsymbol{J}'_{\nu'} + \boldsymbol{J}_\nu) \quad . \quad . \quad (34)$$

In § 11 we shall arrive at a very clear picture of the meaning of this boundary condition which holds for the normal component of the vector $\boldsymbol{D}$.

In a precisely similar way we get for the normal component of the magnetic vector $\boldsymbol{B}$ the simpler boundary condition :

$$\frac{\partial}{\partial t}(\boldsymbol{B}'_{\nu'} + \boldsymbol{B}_\nu) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

§ 11. The most important characteristic of Maxwell's field equations (31) is that they can be integrated in a simple way; this leads to a principle of fundamental importance in electrodynamics. To obtain this directly in a general form we imagine an electromagnetic field in a system consisting of any arbitrary number of bodies in contact and fix our attention on an arbitrarily great volume of any form, which we mark off by an ideal closed surface (shown dotted in Fig. 1). Various bodies are situated in this space, which are separated from each other by definite surfaces of separation (depicted by thick lines in Fig. 1).

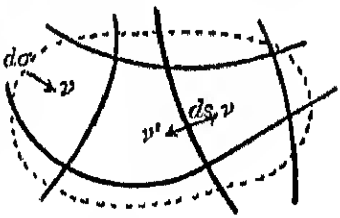

Fig. 1.

We now form the divergence, in the sense of II, 66, for each side of the *electrical* field equations (31*a*). This gives :

$$\operatorname{div} \dot{\boldsymbol{D}} = -4\pi \operatorname{div} \boldsymbol{J} \quad . \quad . \quad . \quad . \quad . \quad (36)$$

We multiply both sides by the space-element $d\tau$ and integrate over the whole space under consideration :

$$\int d\tau \,.\, \operatorname{div} \dot{\boldsymbol{D}} = -4\pi \int d\tau \,.\, \operatorname{div} \boldsymbol{J} \quad . \quad . \quad . \quad (37)$$

The expression on the left-hand side can immediately be expressed as a differential coefficient with respect to the time t, while the expression on the right-hand side can be transformed in the manner of (25) into a surface integral. But it must be remembered that the vector $\boldsymbol{J}$ is not continuous and hence the transformation (25) cannot simply be applied to the whole of the space in question. Rather, we first divide the space-integral on the right-hand side of (37) into the sum of a number of space-integrals, each of which refers to a single homogeneous body of the system, and we then perform the transformation into a surface integral for each body separately. Then the various surface integrals contain, besides the external ideal surface, also the internal real surfaces of separation of each pair of bodies. A distinction asserts itself in that the elements ($d\sigma$) of the ideal surface occur only once, whereas every element (ds) of a real surface of separation occurs twice, since it occurs as a surface element for two different bodies. In view of this, the equation (37) becomes:

$$\frac{\partial}{\partial t}\int d\tau \,.\, \mathrm{div}\, \boldsymbol{D} = 4\pi \int d\sigma \boldsymbol{J}_\nu + 4\pi \int ds(\boldsymbol{J}'_{\nu'} + \boldsymbol{J}_\nu)$$

if we combine the two terms which refer to a definite element ds into one and distinguish the two bodies adjoining ds by an accent.

By (34) we can now replace the integral over the surfaces of separation by a differential quotient with respect to the time and so write the last equation in the form:

$$\frac{\partial}{\partial t}\left\{\int d\tau \cdot \frac{\mathrm{div}\, \boldsymbol{D}}{4\pi} + \int ds \cdot \frac{\boldsymbol{D}'_{\nu'} + \boldsymbol{D}_\nu}{4\pi}\right\} = \int d\sigma \boldsymbol{J}_\nu \quad . \quad (38)$$

This expresses that the time change of the whole quantity in the brackets:

$$\int d\tau \cdot \frac{\mathrm{div}\, \boldsymbol{D}}{4\pi} + \int ds \cdot \frac{\boldsymbol{D}'_{\nu'} + \boldsymbol{D}_\nu}{4\pi} = e \quad . \quad . \quad (39)$$

is dependent on events at the external surface of the space in question. If the surface integral on the right-hand side

is equal to zero, which by (29) is so, for example, if $\kappa = 0$ for all media through which the surface passes, the above quantity remains constant, The quantity e defined by (39) is called the *total quantity of electricity* or *the electric charge* contained in the system of bodies under consideration and the law expressed by (38) is called the *Principle of the Conservation of Electricity.*

As we see from (39), the electric charge exists in two different forms, as a space charge and as a surface charge. The "space density of charge" is:

$$\frac{\operatorname{div} \boldsymbol{D}}{4\pi} = k \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

whereas the "surface density of charge" is:

$$\frac{\boldsymbol{D}'_{\nu'} + \boldsymbol{D}_{\nu}}{4\pi} = h \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

The vector $\boldsymbol{D}$ defined by (28) is called the "electric induction" or "electrical excitation." Accordingly, the spatial density of charge is proportional to the "space or volume divergence," and the surface density of charge is proportional to the "surface divergence"—that is, to the abrupt change of the normal component of the electric induction.

The total charge e of the system of bodies may, by (39), be represented by an integral over the outer surface of the system. For if we transform the space-integral in (39) in exactly the same manner as was done above with the space-integral on the right-hand side of (37), bearing in mind that the vector $\boldsymbol{D}$ at a surface of separation of two different bodies is in general discontinuous, two surface integrals result: one over the surfaces of separation, which is exactly equal and opposite to the one which occurs in (39) and cancels with it; the other integral is over the outer surface and runs:

$$-\int d\sigma \cdot \frac{\boldsymbol{D}_{\nu}}{4\pi} = e \quad . \quad . \quad . \quad . \quad . \quad . \quad (42$$

The relationship expressed by this equation between the total quantity of electricity within the space in question and the surface integral over the normal component of the electric induction is of exactly the same kind as the relation deduced in II, § 64 between the sum of the intensities of all the sources and sinks contained in a volume of liquid and the flux through the surface of the volume (Gauss's equation). Hence the product $d\sigma \cdot \boldsymbol{D}_\nu$ is also called the "flux of induction" through the surface-element $d\sigma$ in the direction of the normal ν. The total flux of induction *outwards* through an arbitrarily closed surface thus always gives the total charge in the space bounded by the surface.

Further, by (38) the time change of the total charge e of the space in question amounts to:

$$dt \cdot \int d\sigma \cdot \boldsymbol{J}_\nu \qquad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

in the time dt.

Hence this quantity is called the total quantity of electricity which flows into the space through all the surface-elements in the time dt. Since the quantity of electricity:

$$dt \cdot d\sigma \cdot \boldsymbol{J}_\nu \qquad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

flows through the surface-element $d\sigma$, the vector $\boldsymbol{J}$ defined by (29) is called, analogously to the energy-flux vector $\boldsymbol{S}$ in (5), the "electric current density"; and the constant κ in (29) is called the "specific electric conductivity" of the medium in question. For a vacuum $\kappa = 0$, by § 8, and hence $\boldsymbol{J} = 0$. Such substances are called "insulators," all others are called "conductors." If κ can be set equal to infinity for a substance, it is called a "perfect conductor."

It is obvious that the introduction of the concept of the electric charge, the electric current and the electric conductivity allows us to form a clear picture of the principle of the conservation of electricity. For example, by introducing the surface density h from (41), we can write the boundary condition (34) in the form:

$$\frac{\partial h}{\partial t} = -\left(\boldsymbol{J}'_\nu + \boldsymbol{J}_\nu\right) \qquad . \quad . \quad . \quad . \quad . \quad (45)$$

take it as meaning that the abrupt change which the nal component of the electric current density undergoes surface determines the time change of the amount of ge which resides on the surface. The relations for netism corresponding to the principle of the conserva- of electricity are much simpler. For since in this the term relating to the conduction current does not r, these relations reduce to:

$$\frac{\partial}{\partial t} \operatorname{div} \boldsymbol{B} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

h can be obtained directly from (31b), and:

$$\frac{\partial}{\partial t} (\boldsymbol{B}'_{\nu'} + \boldsymbol{B}_{\nu}) = 0 \, . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

ı (35). That is, the magnetic charge, both the space the surface charge, is everywhere invariable in time. ce $\boldsymbol{B}$ denotes the "magnetic induction" defined by

12. For homogeneous bodies, to which we shall rict our attention in the sequel, the equation (36) its quite generally of integration with respect to the . For since ϵ and κ are constant in space, they can placed in front of the div-sign, so that we get, in equence of (28), (29) and (21):

$$\frac{\partial k}{\partial t} = -\frac{k}{T}$$

h, when integrated, gives:

$$k = \text{const.}\, e^{-\frac{t}{T}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

t is, in a homogeneous substance the density of electric ge in space decreases with the time under all circum- ces. It is only in the limiting case when the time of xation T becomes infinitely great that it remains tant. Thus if at any time no space charge was ent, none can ever present itself. In view of this rk we shall in the sequel everywhere assume the

density of charge k in space to be zero everywhere. From (40) it then follows generally that:

$$\operatorname{div} \boldsymbol{D} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

In homogeneous bodies electricity is always found to accumulate at the bounding surfaces.

We also obtain further from (28) and (29) that:

$$\operatorname{div} \boldsymbol{E} = 0 \text{ and } \operatorname{div} \boldsymbol{J} = 0 \quad . \quad . \quad . \quad (50)$$

Thus the electric current behaves analogously to a flowing incompressible liquid (cf. II (259)). But we must remember that the space-density of electricity is equal to zero. This, of course, gives rise to no contradiction, since the divergence conditions (49) and (50) are certainly compatible with a finite value of $\boldsymbol{J}$.

Concerning magnetism, we have already seen in (46) and (47) that neither the space nor the surface density of charge can change with the time, and hence in all bodies which have ever been non-magnetic it must continue to remain zero. Hence we specialize our equations still further by assuming generally that:

$$\operatorname{div} \boldsymbol{B} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

and:

$$\boldsymbol{B}'_{\nu'} + \boldsymbol{B}_{\nu} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

That is, in the case of magnetic induction both the volume divergence and the surface divergence vanish. Hence, in analogy with the general equation (42) the flux of magnetic induction through any closed surface is always equal to zero, or:

$$\int \boldsymbol{B}_{\nu} d\sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

The above may be regarded as giving a complete account of the foundations of the whole theory of electricity and magnetism.

PART TWO

STATICAL AND STATIONARY STATES

CHAPTER I

ELECTROSTATIC FIELD WITHOUT CONTACT POTENTIALS

§ 13. Turning to the applications of the systems of equations that have been derived, we shall now, in order gradually to obtain a survey of the vast field of phenomena, start by treating some simple cases, first of all *statical states*. We call an electromagnetic field statical if the state of the medium filled with the field nowhere undergoes changes with the time. Hence in a statical field all differential quotients with respect to the time t must vanish, from which it follows by (31) that:

$$0 = c \operatorname{curl} \boldsymbol{H} - 4\pi \boldsymbol{J} \quad . \quad . \quad . \quad . \quad (54)$$

$$0 = \operatorname{curl} \boldsymbol{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

But these conditions are not sufficient to define the statical state. For so long as the quantity $\kappa \boldsymbol{E}^2$ in (24) differs from zero at any point of the medium, heat is generated there, and so a change of state occurs in the medium. Hence the condition for a statical state demands that in the product $\kappa \boldsymbol{E}^2$ either the factor κ or the factor $\boldsymbol{E}$ must vanish everywhere. In other words, a finite electrostatic field is possible only in an insulator ($\kappa = 0$), whereas if there is the slightest trace of conductivity the electric intensity of field in the statical field is zero, that is:

$$\boldsymbol{E} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

If the quantity $\kappa \boldsymbol{E}^2$ vanishes then the quantity $\kappa \boldsymbol{E} = \boldsymbol{J}$ also vanishes everywhere, and hence it follows from (54) that:

$$0 = \operatorname{curl} \boldsymbol{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

These are the fundamental equations for stationary fields. As we see, they subdivide into those which contain only electrical quantities and those which contain only magnetic quantities; or, in other words, electrostatic and magnetostatic fields have quite independent laws. They simply become superposed without influencing each other to the slightest degree. Hence we can treat electrostatics quite independently of magnetostatics.

Although these statements are very simple and easy to understand, they nevertheless lead to a very remarkable consequence. If we imagine an electrostatic and a magnetostatic field to be present simultaneously in a medium, then according to (26) a definite finite energy flux occurs everywhere, and its magnitude and direction depend essentially on the behaviour of the electric and the magnetic field to each other. Here then we actually have a process which does depend on the interaction between the electric and the magnetic field. It is true that there is no physical significance attached to this process, for the energy flux nowhere effects a change in the energy that is present; rather, just as much energy flows into every part of space as flows out. But this consequence, deduced from Poynting's theorem, is nevertheless somewhat striking and must for the present be regarded as a result that is not so far supported by fact and so constitutes an unnecessary burden for the theoretical ideas; yet it cannot be discarded without endangering the whole structure of Maxwell's theory.

§ 14. We shall now first obtain the laws of *electrostatics*. From (55) it follows that the field intensity $\boldsymbol{E}$ has a potential (I (121)):

$$\boldsymbol{E} = -\operatorname{grad}\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

which reduces the whole problem to investigating a single function. The electric potential ϕ everywhere satisfies the differential equation (50):

$$\Delta\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

and also the boundary condition (11):

$$\frac{\partial(\phi' - \phi)}{\partial\lambda} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

where $d\lambda$ denotes an element of length of any curve lying in the common bounding surface of two substances. Integrating with respect to λ, we get:

$$\phi' - \phi = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (61)$$

that is, the potential difference at the bounding surface of two substances has the same value at all points of the surface. This quantity is called the *contact potential* of the two substances. In general, it differs from zero and depends on the chemical constitution and the temperature of the two substances. In the present chapter we shall disregard contact potentials entirely and so shall consider the potential function as *continuous* throughout. Moreover, the electric potential is *uniform*, since the relation (55) holds everywhere in space (cf. II, § 69).

According to (56) the potential function ϕ is constant at all points inside any conductor. Its absolute value has no physical meaning, since in all the preceding equations only differences and differentials of ϕ occur. Hence we are at liberty to fix the value of ϕ at any arbitrary point as we please. It is usual to set ϕ equal to zero for points at an infinite distance from the electrically charged bodies, where they no longer produce appreciable effects.

If the potential function ϕ is known, we get by (41) and (28) for the density of electric charge at the common bounding surface of two substances:

$$-4\pi h = \epsilon\frac{\partial\phi}{\partial\nu} + \epsilon'\frac{\partial\phi'}{\partial\nu'} \quad . \quad . \quad . \quad . \quad . \quad (62)$$

If the substance indicated by an accent is a conductor, then $\frac{\partial\phi'}{\partial\nu'} = 0$ and:

$$4\pi h = -\epsilon\frac{\partial\phi}{\partial\nu} = \epsilon E_\nu \quad . \quad . \quad . \quad . \quad . \quad (63)$$

Thus the density of charge is proportional to the intensity of field in the insulator and has the same sign as that component of it which is directed towards the insulator.

This also gives us for the total charge on a conductor which borders on an insulator whose dielectric constant is ϵ :

$$e = \int h d\sigma = -\frac{\epsilon}{4\pi}\int \frac{\partial \phi}{\partial \nu} d\sigma \quad . \quad . \quad . \quad . \quad (64)$$

which agrees with the general equation (42); but we must observe that ν is there directed towards the interior of the insulator and so denotes the normal directed towards the space outside the conductor.

This total quantity e of the conductor is a constant quantity which is characteristic of its state. For by § 11 it does not change its value so long as the conductor is surrounded by the insulator, no matter how important the changes that occur in the field may be. If $e = 0$ without $h = 0$ the conductor is said to be charged by "influence" or "induction."

§ 15. A striking picture of the properties of an electric field is obtained if we introduce equipotential surfaces (*niveaux*) ϕ = const. and the lines of forces which intersect them normally and which have already been described in detail in I, § 40. On account of (56) there are no lines of force at all in the interior of a conductor, but only in insulators. The surfaces of conductors are equipotential surfaces and hence the lines of force at all points of a conducting surface are normal to that surface. At the points with a positive charge-density h the lines of force pass, by (63) from the conductor to the insulator; at the points with a negative charge-density they pass from the insulator to the conductor; that is, a line of force begins at a positive and ends at a negative charge. Since a line of force always goes from higher to lower values of the potential, it can never end on the same conductor as it started from. Thus it either ends at the surface of another conductor at lower potential or runs off to in-

finity. We can reduce the latter case to the former case by imagining all points at infinity to be conductively connected by having an infinitely great conducting spherical surface surrounding the whole finite field in question. This in no way alters the field. For since the potential function vanishes at infinity, the condition ϕ = const. is in any case already fulfilled on this spherical surface.

An infinitely thin tube of force passes from every positively charged infinitely small surface-element $d\sigma$ of a conductor to the insulator (cf. II, § 61). Its form is intimately connected not only with the direction, but also with the magnitude of the intensity of field. For by (63) we have :

$$hd\sigma = \frac{1}{4\pi}\epsilon E_\nu d\sigma = \frac{1}{4\pi}D_\nu d\sigma \quad . \quad . \quad . \quad (65)$$

This quantity, which represents the amount of electricity which is cut out of the surface of the conductor by the tube of force, has a constant characteristic significance for the whole course of the tube of force so long as it encounters no charge. If we now take $d\sigma$ to stand for any arbitrary cross-section of the tube in any direction and at any point of the field, ν being its normal, then the quantity (65) represents the flux of induction through the cross-section $d\sigma$, except for the factor 4π in the denominator. According to the general equation (42) the flux of induction outwards through a closed surface which encloses no electric charge is equal to zero, and since we can always form a closed surface from two arbitrary cross-sections of the tube and the tube wall, since there is flux of induction through the tube wall, it follows from (42) that so long as the tube intersects no charges the flux of induction in a definite direction remains constant through any cross-section of the tube. This is completely analogous to the case of the current intensity of an incompressible liquid in II (328), and with the momentum of a vortex filament in II (317).

Where the tube narrows the induction and with it the intensity of field increase, and vice versa. The flux of induction also remains the same at cross-sections where the tube passes from one insulator to another and consequently makes a bend (§ 17), provided that no charge resides on the surface of separation—that is, so long as $h = 0$. The flux of induction is broken off at the end of the tube. It is compensated by the negative charge in which the tube ends; this negative charge is just as great as the positive charge (65) in which it begins. The fact that these two charges are exactly equal and opposite results immediately from applying the general equation (42) to the whole tube of force, if we imagine it closed by two cross-sections which lie entirely in the two bounding conductors. The flux of induction through the closed surface is then equal to zero and hence the algebraic sum of the quantities of electricity in its interior is also equal to zero.

If we assume all of the charges $hd\sigma$ from which all the tubes of force start or in which they all end to be of the same magnitude, then the number of the tubes that start from them, those ending in them being counted as negative, gives a measure, both as regards magnitude and sign, of its total charge e. Hence such tubes are called "unit tubes of force." In graphical representation every unit tube of force is most conveniently depicted by a single *line* of force which runs, say, along the middle of the tube. We can then state that the number of lines of force that start from a conductor gives the flux of induction through its surface and hence the total charge residing on it, both in magnitude and in sign.

If an insulating medium in which an arbitrary number of charged conductors are situated is completely surrounded by a single conductor, then an amount of electricity collects on the inner surface of this conductor which is exactly equal to the total charge of all the enclosed conductors, but of opposite sign. For we can place around the whole insulating medium a closed

surface which lies completely in the substance of the surrounding conductor so that no flux of induction and no line of force passes through it. Hence the total charge enclosed by this surface is equal to zero, or, as we say, a quantity of electricity equal and opposite to the charge on the inner surface of the surrounding conductor is "bound" by the charged conductors. The same result follows if we consider that every positive charge which arises anywhere in the insulator forms the beginning of a tube of force which marks off at its other end an equally great negative charge. A quantity of electricity which represents the difference between the total charge of the conductor and the charge on its inner surface collects on (is "induced" on) the outer surface of the surrounding conductor.

If we make the potential of the surrounding conductor zero by connecting it with an infinitely distant conductor or as we say with the "earth," the charge on the outside flows to earth and the whole of the external space becomes electrically neutral. For since a line of force cannot end on the same conductor as that on which it begins, no lines of force at all can exist in the external space. The electric field in the interior is, however, quite independent of whether the surrounding conductor is connected to earth or whether it is insulated and then has a charge communicated to it. For the conditions of the internal electrostatic field are not affected by the absolute value of the potential of the surrounding conductor (§ 14). Hence we say that the surrounding conductor exerts a "screening action"; this is intended to mean that the electrostatic fields on both sides of the screen act quite independently of each other. A plane conducting surface may also be used as an electric screen if it is taken of sufficient area to collect all the lines of force that run from the field on the one side in the direction of the other side so that no lines of force can pass round the screen into the other field.

§ 16. Turning now to special cases, we first remind

ourselves that every particular solution of Laplace's differential equation (59) represents a definite electrostatic field which comes into being when the corresponding special boundary conditions are fulfilled. The boundaries of the field are formed by the surfaces of the conductors—that is, by surfaces of constant potential. Conversely, if we have any definite function ϕ, which is a solution of (59), we may regard any arbitrary surface ϕ = const. as the surface of a conductor having this value for the potential, and this conductor would produce in its neighbourhood the field represented by the potential function ϕ. All that we are then concerned with is whether the form of this surface is of interest as a conducting surface.

The simplest particular solution of $\Delta\phi = 0$ is a function ϕ which is linear with respect to the rectilinear co-ordinates x, y, z. Corresponding to it we have, according to (58), a *homogeneous* electrostatic field. If we take the direction of the field-strength as the x-axis, ϕ depends only on x, and we have:

$$\phi = ax + b \qquad (66)$$

This field is realized if two equipotential surfaces—say $x = 0$ and $x = D$—which form the boundaries of the field are imagined as surfaces of conductors. The system then forms a *plane condenser*. The electric field is determined by the values of ϕ at the two bounding planes (plates, layers). Let them be $\phi = \phi_0$ for $x = 0$ and $\phi = 0$ for $x = D$, the second plate being assumed to be connected to earth. Then (66) becomes:

$$\phi = \phi_0\left(1 - \frac{x}{D}\right) \qquad (67)$$

The lines of force run parallel to the positive x-axis and the intensity of field or the potential gradient is, by (58):

$$E_x = -\frac{\partial\phi}{\partial x} = \frac{\phi_0}{D} \qquad (68)$$

By (63) the density of charge on the non-earthed plate $x = 0$ is :

$$h = -\frac{\epsilon}{4\pi}\frac{\partial\phi}{\partial x} = \frac{\epsilon}{4\pi}\cdot\frac{\phi_0}{D} \quad . \quad . \quad . \quad . \quad (69)$$

On the earthed plate facing it is the opposite charge.

The ratio of the total charge on a plate to the potential difference between the plates or the charge which produces unit difference of potential between the plates is called the electric "capacity" of the condenser. Thus if F denotes the area of the surface of a plate the capacity is :

$$C = \frac{h \cdot F}{\phi_0} = \frac{\epsilon F}{4\pi D} \quad . \quad . \quad . \quad . \quad (70)$$

That is, the capacity of a plane condenser is proportional to its surface, inversely proportional to the distance between its plates and proportional to the dielectric constant of the insulator (air, glass). Since the formula (66) holds only for infinitely extended equipotential surfaces, the equation (70) requires a correction, the so-called "edge" correction, when applied to a finite surface F. This correction becomes the less important the greater F is.

If a conductor which is insulated and uncharged is introduced into a homogeneous field, it disturbs the field in a certain way which is in general difficult to calculate, but a general idea of the special features of the disturbance can be gained with the help of the laws governing lines of force. For since the lines of force everywhere start from or arrive at the conductor in a direction normal to its surface, the conductor attracts the lines of force in its neighbourhood to itself (Fig. 2), whereas on the other side it repels an equal number of lines of force. The points of arrival form the seat of negative charges, the points of departure or repulsion form the seat of positive charges. On each side there will be a definite *singular* line of force which will pass at such a distance from the conductor that it will just meet it and then leave it again at the same

point (the fact that its vertex encounters the conductor perpendicularly cannot be shown clearly in the figure). At a singular point A of this kind the charge density is equal to zero; it lies in the so-called neutral zone [1] of the conductor which separates the positive from the negative charges. The more distant lines of force of the field link up continuously (laterally) with the singular lines of force, the curves showing first greater and then gradually smaller depressions until the influence of the disturbing conductor at last vanishes.

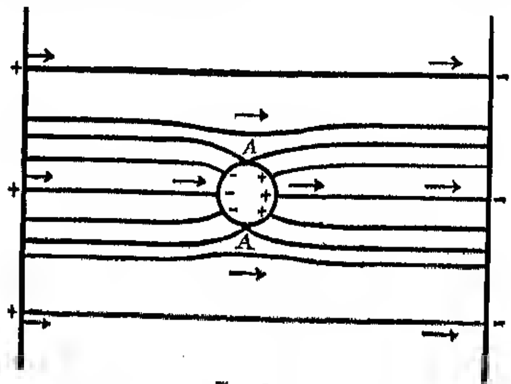

FIG. 2.

The lines of force also give a certain general picture of the relationship between the magnitude of the charge-density and the form of the surface of the conductor. For where the surface is strongly bent outwards the lines of force diverge outwards strongly—that is, the tubes of force rapidly contract as they approach a conductor—the intensity of field therefore increases considerably and the density of charge has a high value. But wherever the surface of the conductor is strongly curved inwards—for example, when it exhibits a hollow—the lines and tubes of force diverge as they approach the conductor, the intensity of field rapidly decreases and the density of charge at the surface is very small.

[1] Also called "lines of no electrification."

Limiting cases are given by projecting points or infinitely sharp hollows, at which the density of charge is infinite and zero respectively. We may regard as an illustration of this Fig. 11 in II, § 63, where the right angles denote the surfaces of the conductors and the curves denote the lines of force in the insulator. The point A denotes a very acute angle, the point B a very obtuse angle.

§ 17. Let us next consider the conditions at the boundary surface of two insulators. The boundary condition is given by (62) in this case, the density of charge h at every point of the boundary surface having a definite invariable value. From this it follows that the direction of the line of force at the boundary surface which is determined by the ratios of the field components is discontinuous—that is, that the lines of force undergo refraction in passing through this surface; the extent of the refraction depends essentially on the density of charge h. If we assume $h = 0$, which transforms (62) into:

$$\epsilon \frac{\partial \phi}{\partial \nu} + \epsilon' \frac{\partial \phi'}{\partial \nu'} = 0 \quad . \quad . \quad . \quad . \quad . \quad (71)$$

the law of refraction assumes a simple form. For if we resolve the intensity of field on each side of the boundary surface into its tangential and its normal components, the former is continuous, whereas the latter, by (71), changes in inverse proportion to the dielectric constants. From this it follows firstly, that the "refracted" line of force lies in the plane defined by the incident line of force and the normal to the surface, and, secondly, that if α and α' denote the acute angles made with the normal by the two lines of force the following relation holds:

$$\tan \alpha : \tan \alpha' = \epsilon : \epsilon' \quad . \quad . \quad . \quad . \quad (72)$$

For the tangents of these two angles are inversely proportional to the normal components of the intensities of field. Thus, the greater the dielectric constant of an insulator, the greater is the angle between the normal and the direction of the line of force that penetrates into it

(that is, the angle of refraction). In the limiting case $\epsilon = \infty$ we get by (71) that $\frac{\partial \phi}{\partial \nu} = 0$ at the whole surface of the insulator and hence by the theorem proved in II. § 71, the value of ϕ is constant in the whole of the interior of the insulator. Or, in other words, an insulator with an infinitely great dielectric constant behaves in an electrostatic field like a conductor; no line of force enters it. This does not, of course, allow us to identify such an insulator entirely with a conductor. For there is always the essential difference between them that an electric current can pass through a conductor, but not through an insulator.

§ 18. A particular solution of Laplace's equation (59) which is of fundamental importance is Newton's gravitational potential (I (125), or (126)) for the case where the reference-point (*Aufpunkt*) lies outside the acting masses. We shall therefore now set the electrostatic potential equal to the potential of certain fictitious positive or negative masses suitably distributed and acting according to the law of gravitation. Since Laplace's equation holds everywhere in an electrostatic field, the fictitious masses cannot have a finite space-density; it therefore suggests itself to imagine them distributed with finite surface-density over the surfaces of conductors and insulators.

In order to avoid confusion with the electric density h, we shall denote the surface-density of these fictitious masses by h' and the gravitational potentials arising from them by ψ. We then have:

$$\psi = \Sigma \frac{\mu}{r} = \int \frac{h' d\sigma}{r} \quad . \quad . \quad . \quad . \quad (73)$$

where r is the positive distance of the reference-point from the surface-element $d\sigma$.

Let us first investigate quite generally the properties of this gravitational potential ψ of masses which are spread over any surface with a definite density h'. According to its definition (73), the function ψ is unique

and is equal to zero at infinity if, as we shall assume, the surface lies entirely in finite regions. The question of the continuity of ψ and of its first differential coefficient can be answered by treating a special case—that of a spherical surface covered uniformly with matter and of radius R. If the reference-point lies outside the sphere, then, by I, § 37, the potential is just as great as if the total mass were concentrated at the centre of the sphere, thus:

$$\psi = \frac{1}{r_0} \cdot \int h' d\sigma = \frac{4\pi R^2}{r_0} h' \quad . \quad . \quad . \quad (74)$$

where r_0 denotes the distance of the reference-point from the centre. If, however, this point lies within the sphere, the potential is everywhere constant, and is just as great as at the centre, thus by (73):

$$\psi = \frac{1}{R} \int h' d\sigma = 4\pi R h' \quad . \quad . \quad . \quad (75)$$

Now if the reference-point moves from outside ($r_0 > R$) through the surface of the sphere $r_0 = R$ into the interior ($r_0 < R$) then, as we see, the expression (74) transforms continuously into the expression (75)—that is, the potential ψ undergoes no abrupt transition at the surface matter. This result can easily be extended to matter distributed on a surface of any form by means of an argument analogous to that used in I, § 33. For if we divide the spherical surface into two parts by marking off around the point A where the reference-point passes through the surface (Fig. 3) a small disc-shaped region on the sphere, which we denote by 1 (shown shaded in the figure), whereas all the remainder is denoted by 2, then we have:

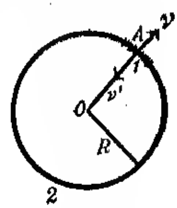

FIG. 3.

$$\psi = \psi_1 + \psi_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

And since ψ and ψ_2 are both continuous—the latter because for the mass 2 the reference-point always remains external

to it during the displacement—so also is ψ_1, the potential of the small disc. But this proves that ψ is continuous when the reference-point passes through matter distributed over a surface of *any* shape. For, however the surface may be constituted, it is always possible to mark off on it a little disc of this kind around the place at which the point of reference passes through, and the remaining part of the surface can cause no discontinuity because the reference-point always remains external to it. Hence we have generally for any arbitrary surface on which matter is distributed:

$$\psi' = \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

where ψ and ψ' denote the values of ψ on both sides of the surface.

An exactly similar argument gives us our result for the first differential coefficient of ψ. Considering first the tangential components, we get by differentiating (77) with regard to any direction λ that lies in the surface:

$$\frac{\partial \psi'}{\partial \lambda} = \frac{\partial \psi}{\partial \lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

So the tangential differential coefficients are continuous. To find the relation between the normal differential coefficients we cannot, of course, differentiate equation (77) in the direction of the normal ν of the surface, because this equation holds only for the surface itself—that is, for a special value of ν, but rather we again start from the simple case of the spherical surface over which matter is uniformly distributed, bearing in mind that we may in this case set $\nu = r_0$. For an external reference-point we then get from (74) that:

$$\frac{\partial \psi}{\partial \nu} = \frac{\partial \psi}{\partial r_0} = -\frac{4\pi R^2}{r_0^{\,2}} h'$$

and if the external point lies just outside the surface:

$$\frac{\partial \psi}{\partial \nu} = -4\pi h' \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

On the other hand, we get for an internal reference-point, from (75) that:

$$\frac{\partial \psi}{\partial \nu} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

Thus the differential quotient $\frac{\partial \psi}{\partial \nu}$ is discontinuous at the spherical surface and undergoes an abrupt change of amount $-4\pi h'$ as the reference-point passes through the surface. That is, if ψ denotes the potential in the outer space, ψ' that in the inner space, ν and ν' the outwardly and inwardly directed normals, we have:

$$\frac{\partial \psi}{\partial \nu} + \frac{\partial \psi'}{\partial \nu'} = -4\pi h' \quad . \quad . \quad . \quad . \quad (81)$$

To extend this result to matter distributed over a surface of any form whatsoever, we again distinguish the two parts 1 and 2 of the spherical surface. Then we have generally, by differentiating (76):

$$\frac{\partial \psi}{\partial \nu} = \frac{\partial \psi_1}{\partial \nu} + \frac{\partial \psi_2}{\partial \nu} \quad . \quad . \quad . \quad . \quad . \quad (82)$$

Likewise:

$$\frac{\partial \psi'}{\partial \nu'} = \frac{\partial \psi'_1}{\partial \nu'} + \frac{\partial \psi_2'}{\partial \nu'} \quad . \quad . \quad . \quad . \quad . \quad (83)$$

But we have:

$$\frac{\partial \psi_2}{\partial \nu} + \frac{\partial \psi_2'}{\partial \nu'} = 0$$

for $\frac{\partial \psi_2}{\partial \nu}$ is continuous, since the reference-point lies outside the part 2 of the surface.

Accordingly, by adding (82) and (83) and bearing in mind (81), we get:

$$\frac{\partial \psi_1}{\partial \nu} + \frac{\partial \psi'_1}{\partial \nu'} = -4\pi h' \quad . \quad . \quad . \quad . \quad (84)$$

From this we see that the abrupt change in the value of the differential coefficient $\frac{\partial \psi}{\partial \nu}$ of the potential of the

spherical surface on which the matter is distributed arises exclusively from the potential of the small disc around A. From this it follows that the amount of the change due to the transition does not depend on how the matter outside the disc is distributed. Hence if we have a material shell of any arbitrary shape and distribution of density, and if we allow the reference-point to pass through the surface at the point A on it, the normal differential coefficient of the potential ψ undergoes the change:

$$\frac{\partial\psi}{\partial\nu}+\frac{\partial\psi'}{\partial\nu'}=-4\pi h' \quad . \quad . \quad . \quad . \quad (85)$$

where h' denotes the density of the surface layer at the point of transition A—that is, the surface density of a small disc described about A.

We can go a step further still in our deductions. For in equation (84) we have:

$$\frac{\partial\psi_1}{\partial\nu}=\frac{\partial\psi_1'}{\partial\nu'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

This is so because in the case of the small disc there is no difference between outside and inside; rather the directions ν and ν' are completely equivalent and interchangeable. From this it follows that:

$$\frac{\partial\psi_1}{\partial\nu}=\frac{\partial\psi'_1}{\partial\nu'}=-2\pi h' \quad . \quad . \quad . \quad . \quad (87)$$

This quantity has a clear physical meaning. For since the gradient of the potential equals the force with which unit mass at the reference-point is attracted or repelled by the acting mass, (87) represents the force which the small disc exerts on a point-mass right at its surface. This force depends only on the density h' of the superficial distribution of mass, and not on the size of the disc; rather, it retains its value no matter how small we assume the size of the disc to be. This is, of course, explained by the fact that the dimensions of the disc are still always infinitely great compared with the distance of the point of reference from the disc.

In this way we also see why the force exerted by matter distributed over a spherical surface suddenly sinks from a finite value to zero when the point moves from outside through the surface into the interior. For so long as the point-mass is still outside the surface the small disc 1 acts in the same direction as the remaining mass 2. But as soon as the point passes through the surface, the direction of the force which starts from the disc becomes reversed, whereas the other force preserves its direction. Thus the two forces then act in opposite directions and compensate each other.

§ 19. Through the results of the preceding investigation we have now advanced sufficiently far to enunciate and prove the following theorem: in any arbitrary electrostatic field the potential ϕ is identical with the gravitational potential ψ of the fictitious masses which are distributed over the surfaces of conductors and insulators with the surface density:

$$h' = -\frac{1}{4\pi}\left(\frac{\partial\phi}{\partial\nu} + \frac{\partial\phi'}{\partial\nu'}\right) = \frac{E_\nu + E'_{\nu'}}{4\pi} \quad . \quad . \quad (88)$$

We prove this by showing that the function:

$$\psi - \phi = \phi_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

where ψ is defined by (73) and (88), vanishes identically. For the function ϕ_0 has the following properties. It satisfies Laplace's differential equation, because both ψ and ϕ do so separately; further, it is unique and continuous, and vanishes at infinity, because this is the case both with ψ and also with ϕ. Finally, the first differential coefficients of ϕ_0 are always continuous, even at the surfaces of conductors and insulators. For in the case of the tangential derivatives we get from (60) and (78):

$$\frac{\partial\phi'_0}{\partial\lambda} = \frac{\partial\phi_0}{\partial\lambda}$$

and for the normal derivatives we get from (85) and (88):

$$\frac{\partial\phi_0}{\partial\nu} + \frac{\partial\phi'_0}{\partial\nu'} = 0.$$

The fact that a function with these properties vanishes identically comes out as a special result in applying a theorem which we shall now prove in its more general form.

A uniform and continuous function of the space co-ordinates, whose first differential coefficients are also continuous and which satisfies Laplace's differential equation is determined within any domain of space if its values are given at all points of the *surface* of this domain. For if f and f' denote two functions which have all these properties and whose values at all points of the surface of this region are equal to each other but are arbitrary elsewhere, then, by II (81), we have for the difference $f_0 = f' - f$:

$$\int\left\{\left(\frac{\partial f_0}{\partial x}\right)^2+\left(\frac{\partial f_0}{\partial y}\right)^2+\left(\frac{\partial f_0}{\partial z}\right)^2\right\}d\tau = -\int f_0\frac{\partial f_0}{\partial \nu}d\sigma - \int f_0 \cdot \Delta f_0 \cdot d\tau \quad (90)$$

In this equation both integrals on the right-hand side are equal to zero, the space integral on account of Laplace's equation, the surface integral because the difference f_0 vanishes at all points of the surface. Consequently the integrand in the positive integral on the left-hand side is zero at all points of space—that is, f_0 is constant throughout, being equal to zero as at the surface.

Hence the same holds for our function ϕ_0, since it vanishes everywhere at infinity.

§ 20. The theorem deduced in the preceding section gives the laws of electrostatics a new and easily pictured meaning, since it makes the electric intensity of field appear like the action at a distance of gravitation, except that now the surface density which must be ascribed to the fictitious masses is not the electric density of charge h, if their gravitational potential is to agree with the electric potential, but rather the quantity h' defined by (88); h' is therefore often called the "free" density of charge, in contrast to the "true" density of charge h. According to (88), the free charge is related to the electric intensity of field E in exactly the same way as the true charge (41)

is related to the electric induction $\boldsymbol{D}$. The advantage of introducing the free charge is that in calculating the potential and the intensity of field we can disregard entirely the dielectric properties of the insulators, so that we deal only with direct action at a distance. Contrary to the true electricity, however, the free electricity does not satisfy the principle of the conservation of energy (§ 11). For example, if we take an insulated and electrically charged conductor out of one insulator and place it in another having a different dielectric constant, the true charge becomes unchanged, whereas the free charge would have a new value.

The connection between the free density of charge h' and the true density of charge h is represented by the equations (88) and (41). In general, it is fairly complicated, but assumes a particularly simple form in the important case of the surface of a conductor. For in this case $E'_{\nu'} = 0$ and consequently:

$$h' = \frac{E_\nu}{4\pi} = -\frac{1}{4\pi} \cdot \frac{\partial \phi}{\partial \nu} \quad . \quad . \quad . \quad . \quad (91)$$

which, when combined with (63), gives:

$$h' = \frac{h}{\epsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

Hence when h is given we can immediately derive from this action at a distance the potential and the intensity of field.

It is interesting to consider the special cases of electrostatic fields treated in § 16 in the light of this theory of action at a distance. If, for example, the reference-point carries unit charge and is situated at any point between two plane condenser plates having the charge-densities $+h$ and $-h$, then, in accordance with the expression (87) for the attraction of a plane disc which is of infinite extent compared with the distance of the reference-point, the unit charge will be attracted by the negatively-charged

plate and repelled by the positively-charged plate, so that the total intensity of field will amount to:

$$E_x = 4\pi h' = \frac{4\pi h}{\epsilon} \quad . \quad . \quad . \quad . \quad (93)$$

which corresponds exactly with the equations (68) and (69).

If we bring into the homogeneous field of the condenser an insulated and uncharged conductor, then, in order that the intensity of field shall be equal to zero everywhere in its interior, charges will have to appear at its surface, whose attraction or repulsion must exactly compensate the attraction or repulsion of the two charges on the condenser—in such a way, of course, that the total charge on the conductor remains equal to zero. Fig. 2 gives a picture of the action of these bound charges.

By (75) the potential for a conducting sphere of radius R and charge e situated in an insulator whose dielectric constant is ϵ is:

$$\phi = \frac{1}{R}\frac{1}{\epsilon}\int h d\sigma = \frac{e}{R\epsilon} \quad . \quad . \quad . \quad . \quad (94)$$

and so the electric capacity of the sphere is:

$$C = \frac{e}{\phi} = R\epsilon \quad . \quad . \quad . \quad . \quad . \quad (95)$$

that is, it is equal to the product of the radius and the dielectric constant. Hence if we introduce the insulated, charged sphere into another dielectric the potential and also the intensity of the field are reduced in the inverse ratio of the dielectric constant.

If, on the other hand, instead of keeping e constant we keep the potential ϕ of the sphere constant while we introduce it into the new insulator—this can be done by connecting the sphere by means of a thin wire with one pole of a very distant constant battery—the charge on the sphere increases proportionally to the dielectric constant; this is effected by a certain quantity of electricity flowing from the battery to the sphere. The connecting wire can always be chosen sufficiently fine not to influence

appreciably the field produced by the sphere (cf. the end of § 25).

§ 21. We shall generalize the last theorems so that they may be applied to any number of conductors of any shape, all of which we shall suppose embedded in a common insulator whose dielectric constant is ϵ. Let us first consider the simple case in which all the conductors except one, denoted by 1, are connected to earth by long, thin wires—that is, are brought to the potential zero—whereas the conductor 1 is kept at the potential 1. The potential function in the insulator is then completely determined by the theorem derived at the end of § 19. We shall call this function f_1.

The lines of force all start from the conductor 1 and end partly on the remaining conductors and partly at infinity. For this reason the charge e_1 on the first conductor 1 is positive, whereas those on the remaining conductors, $e_2, e_3 \ldots$ are negative.

If now, other conditions remaining the same, the conductor 1 is raised from the potential 1 to the potential ϕ_1, then the potential function of the electric field becomes $\phi_1 \cdot f_1$ and the densities of charge on the individual conductors become:

$$h_1 = -\frac{\epsilon\phi_1}{4\pi}\frac{\partial f_1}{\partial \nu_1},\ h_2 = -\frac{\epsilon\phi_1}{4\pi}\frac{\partial f_1}{\partial \nu_2}, \quad . \quad . \quad . \qquad (96)$$

From these expressions we get for the total charges on the conductors:

$$\left.\begin{aligned} e_1 &= \int h_1 d\sigma_1 = -\frac{\epsilon\phi_1}{4\pi}\int\frac{\partial f_1}{\partial \nu_1}d\sigma_1 = \epsilon\phi_1 c_1 \\ e_2 &= \int h_2 d\sigma_2 = -\frac{\epsilon\phi_1}{4\pi}\int\frac{\partial f_1}{\partial \nu_2}d\sigma_2 = \epsilon\phi_1 c_{21} \\ e_3 &= \int h_3 d\sigma_3 = -\frac{\epsilon\phi_1}{4\pi}\int\frac{\partial f_1}{\partial \nu_3}d\sigma_3 = \epsilon\phi_1 c_{31} \end{aligned}\right\} \quad . \qquad (97)$$

where, after what has been said above, the constant c_1 is positive, whereas the remaining constants, $c_{21}, c_{31}, \ldots$, are negative.

The conductor 1 is charged in virtue of its potential ϕ_1;

all the other conductors are charged by induction from the conductor 1. The product ϵc_1 is the capacity of the conductor 1; the other products, taken positively, are called the "coefficients of capacity" of the other conductors. It is important to note that the quantities c, like the function f_1, depend only on the geometrical and not on the electrical conditions of the system. For this makes it possible to reduce the general case so as to apply to the special case here under discussion.

For if we now denote the corresponding potential function for the conductor 2 by f_2, and so forth, the problem of determining the electric field which arises when the conductor 1 is kept at the potential ϕ_1, the conductor 2 at the potential ϕ_2 and so forth, is solved by the potential function:

$$\phi = \phi_1 f_1 + \phi_2 f_2 + \phi_3 f_3 + \quad . \quad . \quad . \quad . \qquad (98)$$

For this function, in the first place, everywhere satisfies Laplace's equation in the insulator, just as all the individual f's do singly, and secondly, at the surface of every conductor it has the prescribed value, since the corresponding function is $f = 1$ there, whereas all the other f's vanish.

From this we then get, by (63), for the densities of charge on the individual conductors:

$$h_1 = -\frac{\epsilon}{4\pi}\left(\phi_1 \frac{\partial f_1}{\partial \nu_1} + \phi_2 \frac{\partial f_2}{\partial \nu_1} + \phi_3 \frac{\partial f_3}{\partial \nu_1} + \ . \ . \ . \ \right) . \qquad (99)$$

and the total charges:

$$\left.\begin{array}{l} e_1 = \epsilon(\phi_1 c_1 \ + \phi_2 c_{12} + \phi_3 c_{13} + \ . \ . \ . \ ) \\ e_2 = \epsilon(\phi_1 c_{21} + \phi_2 c_2 \ + \phi_3 c_{23} + \ . \ . \ . \ ) \\ e_3 = \epsilon(\phi_1 c_{31} + \phi_2 c_{32} + \phi_3 c_3 \ + \ . \ . \ . \ ) \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{array}\right\} \quad . \quad (100)$$

where:

$$c_{12} = -\frac{1}{4\pi}\int \frac{\partial f_2}{\partial \nu_1} d\sigma_1,\ c_{21} = -\frac{1}{4\pi}\int \frac{\partial f_1}{\partial \nu_2} d\sigma_2 \ . \ . \ . \qquad (101)$$

Thus the charges on the conductors are linear functions of their potentials, and are, moreover, proportional to the

dielectric constant of the insulator, as we saw in (95) for the special case of a sphere. Moreover, we always have:

$$c_{12} = c_{21}, \text{ and so forth} \quad . \quad . \quad . \quad (102)$$

This is easily seen by considering the following identity, which arises from II (80):

$$\int\left(\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial y} + \frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial z}\right)d\tau = \\ - \int f_1 \frac{\partial f_2}{\partial \nu_1}d\sigma_1 - \int f_1 \frac{\partial f_2}{\partial \nu_2}d\sigma_2 - \int f_1 \frac{\partial f_2}{\partial \nu_3}d\sigma_3 - \cdot \cdot \cdot$$

The space integral is here to be taken over the whole insulator, whereas the surface integrals stretch over the whole bounding surface of the insulator—that is, over the surfaces of the conductors. Now since the function f_1 has the value 1 at the surface of the conductor 1 but the value 0 at the surface of all the other conductors, the whole sum reduces to the first term, which is nothing other than the coefficient of capacity c_{12}, except for the factor 4π; and on account of the symmetry of the space integral we get the value c_{21} for it in the same way.

If we solve the equations (100) for $\phi_1, \phi_2 \ldots$, we conversely obtain the potentials of the conductors as linear functions of their charges, thus:

$$\left.\begin{aligned} \phi_1 &= \frac{1}{\epsilon}(e_1 c'_1 + e_2 c'_{21} + e_3 c'_{31} + \ldots) \\ \phi_2 &= \frac{1}{\epsilon}(e_1 c'_{12} + e_2 c'_2 + e_3 c'_{32} + \ldots) \\ \phi_3 &= \frac{1}{\epsilon}(e_1 c'_{13} + e_2 c'_{23} + e_3 c'_3 + \ldots) \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad . \quad (103)$$

where the constants c', like the constants c, depend only on the geometrical and not on the electrical conditions of the system in question. Of course we again have $c'_{12} = c'_{21}$. Thus for given charges the potentials are inversely proportional to the dielectric constant, as we have already seen above for the special case of a sphere.

In the same way we can deal with the problem of deter-

mining the electric field when the potentials of some conductors are given and the charges on the remainder are known. It is always simply a matter of solving linear equations, provided that the capacities c are known.

§ 22. We shall now consider several other particular solutions which are interesting for electrostatics. For simplicity we shall from now onwards choose as our insulator a perfect vacuum, so that $\epsilon = 1$. The free charge then coincides with the true charge. We can then generalize the results for other insulators by means of the above theorems.

From the expression (73) we take the special solution:

$$\phi = \frac{e_1}{r_1} + \frac{e_2}{r_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

where r_1 and r_2 denote the distances of the reference-point P from two fixed poles A and B carrying the charges e_1 and e_2. The equipotential surfaces are of the eighth order, and so are fairly complicated, but among them there is one of rather simple form, namely:

$$\frac{e_1}{r_1} + \frac{e_2}{r_2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

This surface is a sphere, as may easily be seen either analytically or geometrically. Of course, if the sphere is to be real, the charges e_1 and e_2 must have opposite signs. The sphere then encloses the pole which has the weaker charge; let this be B. If O denotes the centre and R the radius of the sphere, and if we denote the distances of the two poles from O by a and b, then we get (Fig. 4) from (105) for the two points of intersection of the sphere with the axis ABO the values:

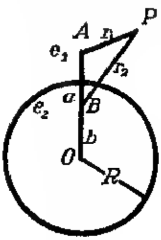

FIG. 4.

$$\frac{e_1}{a+R} + \frac{e_2}{b+R} = 0 \text{ (point of intersection outside } AB \quad (106a)$$

$$\frac{e_1}{a-R} + \frac{e_2}{R-b} = 0 \text{ (point of intersection inside } AB \quad (106b)$$

If the position of the two poles, and hence also the distance $a-b$ between them, and their charges e_1 and e_2 are given, these two equations enable us to find both the radius R of the sphere and also the individual values of a and b—that is, the position of the centre O.

But we can also argue in the reverse direction. For if we consider the sphere to be a conductor and connected to earth, the electric field will not be changed in the slightest. But the sphere then screens its internal space entirely from the external space (§ 15) and the inner and outer fields are totally independent of each other. Hence the case under consideration also contains the solution of the problem of determining the electric field which is formed when a pole A carrying the charge e_1 is situated at the distance a from the centre O of a sphere of radius R which is connected to earth. In the equations (106) we are then to assume e_1, a and R as given and b and e_2 as determined by them. The potential of the field in the external space is then, as before, given by (104). That is, the charge of opposite sign which is held bound on the sphere by the charged pole acts in the external space, just like a pole B which carries the charge e_2 and is at a distance b from the centre of the sphere.

For this reason the point B is also called the "electrical image" of the point A with respect to the sphere. The position of the image B follows from the equations (100) by eliminating e_1 and e_2, and is:

$$b = \frac{R^2}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

This relation is independent of the values of the charges, and is, moreover, reciprocal—that is, the poles A and B can be interchanged. Hence the law (107) which allows the external space to be represented on the internal space and vice versa, is also called the law of "reciprocal radii." From (100) we also get for the charge on the image point B the value:

$$e_2 = -\frac{R}{a} \cdot e_1 = -\frac{b}{R} \cdot e_1 \quad . \quad . \quad . \quad . \quad (108)$$

This completely solves the problem. The surface-density of the charge bound on the sphere comes out from (63) and (104) as:

$$h = -\frac{1}{4\pi}\left(e_1\frac{\partial\frac{1}{r_1}}{\partial r_0} + e_2\frac{\partial\frac{1}{r_2}}{\partial r_0}\right) \quad . \quad . \quad (109)$$

where r_0 denotes the distance of the point of reference P from O, which coincides with the inward normal of the surface of the insulator. To find the total charge on the sphere we must integrate $hd\sigma$ over the whole surface of the sphere. But we get this more easily by remembering that the total charge is determined by the flux of induction through the spherical surface and that this flux of induction remains the same even if the conducting sphere is removed and we imagine instead the pole B carrying a charge e to be present in the vacuum. Thus the charge e_2 is responsible for this flux of induction, and hence e_2 is the total charge bound on the earthed spherical surface.

The preceding calculation also disposes of the case where the sphere is not earthed but is insulated and provided with a given charge. For we need only superimpose on the density of charge h, bound on the sphere by the charge e_1 at the pole A, any other arbitrary uniform positive or negative density of charge to obtain an electrostatic field that corresponds to any desired total charge of the sphere. For example, we have only to distribute the charge $-e$ uniformly over the spherical surface to find the electrostatic field that is produced when an insulated uncharged sphere is placed opposite an electric pole A.

In all the preceding cases we can, of course, exchange the outer space with the inner space—that is, the potential (104) also gives the field in the interior of an evacuated hollow sphere whose surface is earthed and in whose interior there is a pole B carrying the charge e_2. The charge $-e_2$ bound on the spherical surface by e_2 acts in the whole of the interior of the hollow sphere like the image A with the charge e_1.

§ 23. Another method of finding particular solutions of Laplace's equation (59) depends on the introduction of space co-ordinates instead of the rectangular co-ordinates x, y, z. We shall therefore now express the quantity $\Delta\phi$ by new curvilinear co-ordinates, and for this purpose we shall assume x, y, z to be certain given uniform and continuous functions of the three new variables λ, μ, ν.

To arrive at the general formulæ of transformation we form the differentials:

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial \lambda} d\lambda + \frac{\partial x}{\partial \mu} d\mu + \frac{\partial x}{\partial \nu} d\nu \\ dy &= \frac{\partial y}{\partial \lambda} d\lambda + \frac{\partial y}{\partial \mu} d\mu + \frac{\partial y}{\partial \nu} d\nu \\ dz &= \frac{\partial z}{\partial \lambda} d\lambda + \frac{\partial z}{\partial \mu} d\mu + \frac{\partial z}{\partial \nu} d\nu \end{aligned}\right\} \quad . \quad . \quad (110)$$

and consider the nine differential coefficients in them as constant, but the six differentials as variables—that is, we investigate the relationships between the old and the new co-ordinates in the immediate neighbourhood of a definite favoured point $(x, y, z, \lambda, \mu, \nu)$ which we shall call A. We obtain as the square of the distance ds of any infinitely near point P, having the co-ordinates $(x + dx, y + dy, \ldots)$, from the point A the sum of the squares of the three differential expressions (110)—that is, a homogeneous quadratic function of the differentials $d\lambda$, $d\mu$, $d\nu$. For the sake of simplicity we shall, however, now introduce the restrictive assumption that quite generally:

$$\left.\begin{aligned} \frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \mu} + \frac{\partial y}{\partial \lambda}\frac{\partial y}{\partial \mu} + \frac{\partial z}{\partial \lambda}\frac{\partial z}{\partial \mu} &= 0 \\ \frac{\partial x}{\partial \mu}\frac{\partial x}{\partial \nu} + \frac{\partial y}{\partial \mu}\frac{\partial y}{\partial \nu} + \frac{\partial z}{\partial \mu}\frac{\partial z}{\partial \nu} &= 0 \\ \frac{\partial x}{\partial \nu}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \nu}\frac{\partial y}{\partial \lambda} + \frac{\partial z}{\partial \nu}\frac{\partial z}{\partial \lambda} &= 0 \end{aligned}\right\} \quad . \quad . \quad (111)$$

We then have:

$$ds^2 = dx^2 + dy^2 + dz^2 = \frac{d\lambda^2}{\Lambda^2} + \frac{d\mu^2}{M^2} + \frac{d\nu^2}{N^2}$$

where we have used the abbreviations:

$$\frac{1}{\Lambda^2} = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2, \text{ and so forth} \quad . \quad (112)$$

with the restriction that Λ, M, N be positive.

If we now write the equations (110) in the following form:

$$\left.\begin{aligned} dx &= \left(\Lambda\frac{\partial x}{\partial \lambda}\right)\cdot\frac{d\lambda}{\Lambda} + \left(M\frac{\partial x}{\partial \mu}\right)\cdot\frac{d\mu}{M} + \left(N\frac{\partial x}{\partial \nu}\right)\cdot\frac{d\nu}{N} \\ dy &= \left(\Lambda\frac{\partial y}{\partial \lambda}\right)\cdot\frac{d\lambda}{\Lambda} + \left(M\frac{\partial y}{\partial \mu}\right)\cdot\frac{d\mu}{M} + \left(N\frac{\partial y}{\partial \nu}\right)\cdot\frac{d\nu}{N} \\ dz &= \left(\Lambda\frac{\partial z}{\partial \lambda}\right)\cdot\frac{d\lambda}{\Lambda} + \left(M\frac{\partial z}{\partial \mu}\right)\cdot\frac{d\mu}{M} + \left(N\frac{\partial z}{\partial \nu}\right)\cdot\frac{d\nu}{N} \end{aligned}\right\} \quad (113)$$

we can give them a very clear meaning. For if we reflect that the nine constant coefficients, determined by the position of A and enclosed in brackets—we shall call them α_1, α_2, α_3, β_1, β_2, β_3, γ_1, γ_2, γ_3—satisfy, by (112) and (111), the six conditions:

$$\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1, \ldots, \ldots$$
$$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = 0, \ldots, \ldots$$

then the equations (113) may be interpreted, according to I, § 56, as the relationship between the co-ordinates dx, dy, dz and the co-ordinates $\frac{d\lambda}{\Lambda}, \frac{d\mu}{M}, \frac{d\nu}{N}$ of the point P in another similar rectilinear system of reference, which has the same origin A, whose position is defined by the nine direction-cosines in such a way that the co-ordinates x, y, z correspond to the letters α, β, γ and the co-ordinates λ, μ, ν correspond to the numbers 1, 2, 3. We shall assume the latter system also to be right-handed; this can be effected by exchanging λ for μ if necessary. Then all the relations already derived in I, § 146, will hold for the direction-cosines; thus:

$$\Lambda\frac{\partial x}{\partial \lambda} = MN\left(\frac{\partial y}{\partial \mu}\frac{\partial z}{\partial \nu} - \frac{\partial y}{\partial \nu}\frac{\partial z}{\partial \mu}\right), \ldots$$

Further, the converse equations also hold:

$$\frac{\partial\lambda}{\Lambda} = \left(\Lambda\frac{\partial x}{\partial\lambda}\right)\cdot dx + \left(\Lambda\frac{\partial y}{\partial\lambda}\right)\cdot dy + \left(\Lambda\frac{\partial\lambda}{\partial z}\right)\cdot dz, \quad . \quad . \quad .$$

These, when combined with the identities:

$$d\lambda = \frac{\partial\lambda}{\partial x}dx + \frac{\partial\lambda}{\partial y}dy + \frac{\partial\lambda}{\partial z}dz, \quad . \quad . \quad .$$

give us the nine relations:

$$\frac{\partial\lambda}{\partial x} = \Lambda^2\frac{\partial x}{\partial\lambda},\ \frac{\partial\lambda}{\partial y} = \Lambda^2\frac{\partial y}{\partial\lambda}, \quad . \quad . \quad . \qquad (114)$$

by which we can transform from the differential equations with respect to x, y, z to those with respect to λ, μ, ν. From them there also result the following three equations, which correspond with the equations (111):

$$\frac{\partial\lambda}{\partial x}\frac{\partial\mu}{\partial x} + \frac{\partial\lambda}{\partial y}\frac{\partial\mu}{\partial y} + \frac{\partial\lambda}{\partial z}\frac{\partial\mu}{\partial z} = 0, \quad . \quad . \quad . \quad . \qquad (115)$$

which state that the three families of surfaces λ = const., μ = const., ν = const. are everywhere orthogonal to each other. Hence the equations (115) or the equations (111) which are equivalent to them are also called the conditions of *orthogonal substitution*, and the quantities λ, μ, ν are called orthogonal co-ordinates. The points A and P can be regarded as opposite corners in each of two infinitely small rectangular parallelepipeds. The one is bounded by the surfaces:

$$x = \text{const.},\ x + dx = \text{const.},$$
$$y = \text{const.},\ y + dy = \text{const.}, \quad . \quad . \quad .$$

and has edges of length dx, dy, dz. The other is bounded by the surfaces λ = const., $\lambda + d\lambda$ = const., μ = const., $\mu + d\mu$ = const., . . . and has edges of length $\frac{d\lambda}{\Lambda}, \frac{d\mu}{M}, \frac{d\nu}{N}$. Its volume is:

$$d\tau = \frac{d\lambda}{\Lambda}\cdot\frac{d\mu}{M}\cdot\frac{d\nu}{N} \quad . \quad . \quad . \quad . \quad . \qquad (116)$$

To enable us to express the quantity $\Delta\phi$ as simply as possible in the new co-ordinates we use the identity:

$$\int \Delta\phi \, . \, d\tau = -\int \frac{\partial \phi}{\partial n} ds \, . \quad . \quad . \quad . \qquad (117)$$

and apply it to the second of the two infinitely small parallelepipeds. The space-integral on the left then reduces to the single term $\Delta\phi \, . \, d\tau$, where $d\tau$ has the value (116), whereas the surface integral on the right represents a sum of six terms, corresponding to the six faces. The size of the side $\lambda =$ const. is;

$$ds = \frac{d\mu}{M} \cdot \frac{d\nu}{N}.$$

Further the element of length of the inner normal is;

$$dn = \frac{d\lambda}{\Lambda}$$

so that the term to be used in the calculation is:

$$\left(\frac{\partial \phi}{\partial n} ds\right)_\lambda = \frac{\partial \phi}{\partial \lambda} \cdot \frac{\Lambda}{MN} \cdot d\mu d\nu$$

To it there must be added the term referring to the opposite side $\lambda + d\lambda =$ const., which differs from the former only in having $\lambda + d\lambda$ in place of λ and the sign of the inward normal reversed, thus:

$$\left(\frac{\partial \phi}{\partial n} ds\right)_{\lambda + d\lambda} = -\left(\frac{\partial \phi}{\partial \lambda} \cdot \frac{\Lambda}{MN}\right)_{\lambda + d\lambda} \cdot d\mu d\nu.$$

If these two terms are added together we get:

$$-\frac{\partial}{\partial \lambda}\left(\frac{\partial \phi}{\partial \lambda} \cdot \frac{\Lambda}{MN}\right) \cdot d\lambda d\mu d\nu$$

and if this is substituted with the corresponding expressions for μ and ν in (117) we get the desired relation for expressing $\Delta\phi$ in arbitrary orthogonal co-ordinates;

$$\Delta\phi = \Lambda MN\left\{\frac{\partial}{\partial \lambda}\left(\frac{\Lambda}{MN}\frac{\partial \phi}{\partial \lambda}\right) + \frac{\partial}{\partial \mu}\left(\frac{M}{N\Lambda}\frac{\partial \phi}{\partial \mu}\right) + \frac{\partial}{\partial \nu}\left(\frac{N}{\Lambda M}\frac{\partial \phi}{\partial \nu}\right)\right\} \quad (118)$$

§ 24. To take a simple illustration, let us apply the equation of transformation (118) to the ordinary polar co-ordinates:

$$x = r \sin\theta \cos\psi, \; y = r \sin\theta \sin\psi, \; z = r\cos\theta \quad . \quad (119)$$

by identifying r with λ, θ with μ and ψ with ν. The fact that these co-ordinates are orthogonal is seen clearly from the fact that the equations (111) hold. From (112) we then get:

$$\frac{1}{\Lambda} = 1, \; \frac{1}{M} = r, \; \frac{1}{N} = r \sin\theta.$$

All the other relations follow from this. For example, by (114):

$$\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r} \text{ (cf. II, § 65)}$$

and further, by (116): $d\tau = r^2 \sin\theta \, dr \, d\theta \, d\psi$ (cf. I (93)). And from (118) we get:

$$\Delta\phi = \frac{1}{r^2 \sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial}{\partial\psi}\left(\frac{1}{\sin\theta}\frac{\partial\phi}{\partial\psi}\right)\right\} \quad (120)$$

a result which can be obtained only by laborious calculations if we transform directly according to (119).

For the special case in which the potential ϕ depends only on r and not on the angles θ, ψ, we get from (120):

$$\Delta\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\phi}{\partial r}\right) \quad . \quad . \quad . \quad . \quad (121)$$

and Laplace's equation runs:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 0$$

from which we get for the general integral:

$$\phi = \frac{\text{const.}}{r} + \text{const.}$$

which is Newton's potential with a constant added.

§ 25. We shall now make another application of the equation of transformation (118), which will lead us to a new particular solution of Laplace's equation.

We first define a function u of the three co-ordinates x, y, z and three positive constants $a>b>c$ by means of the following equation:

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}-1=0 \quad . \quad (122)$$

As we see, this does not uniquely define u. Rather, since the equation is a cubic, u has three values which in general differ from one another. The fact that these three roots are always real can be seen if we allow u to increase continuously from $-\infty$ to $+\infty$ and investigate the course of the values assumed by the quantity on the left-hand side (which we shall call F) of (122). We get the following table:

For:	We get:
$-\infty < u < -a^2$	$-1 > F > -\infty$
$-a^2 < u < -b^2$	$+\infty > F > -\infty$
$-b^2 < u < -c^2$	$+\infty > F > -\infty$
$-c^2 < u < +\infty$	$+\infty > F > -1$

This shows that the equation $F=0$ has three real roots for u, which we shall call λ, μ, ν, in order of magnitude. Then:

$$+\infty \geqq \lambda \geqq -c^2 \geqq \mu \geqq -b^2 \geqq \nu \geqq -a^2 \quad . \quad . \quad (123)$$

These statements uniquely determine the quantities λ, μ, ν in terms of the rectilinear co-ordinates x, y, z of a point. We call them the *elliptic co-ordinates* of the point, because the surfaces $\lambda=$ const., $\mu=$ const., $\nu=$ const., as we see from (122), are the equations of an ellipsoid, hyperboloid of one sheet and hyperboloid of two sheets. Even if it is not possible to represent λ, μ, ν explicitly in a convenient form as functions of x, y, z yet conversely x, y, z can be expressed in a simple way by means of λ, μ, ν. A very direct method of doing this is as follows. We suppose the

function (122) to be reduced to a common-denominator. The numerator then represents a complete function of the third degree in u, which, by introducing the roots λ, μ, ν, can be written in the form of a product of three linear factors in the following way:

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}-1=\frac{-(u-\lambda)(u-\mu)(u-\nu)}{(a^2+u)(b^2+u)(c^2+u)}. \quad (124)$$

We must be careful to see that the coefficient of u^3 is the same on both sides of the identity.

The meaning of the relation (124) is that it is an identity which holds for *every* value of u, if we regard λ, μ, ν as functions of x, y, z, a, b, c. Hence (124) remains true if we set $u = -a^2$ in it. But on the left-hand side only the first term remains, since it becomes infinite compared with the following terms. So we get:

$$x^2 = \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)} \quad . \quad . \quad (125a)$$

In the same way by cyclic exchange:

$$y^2 = \frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-c^2)(b^2-a^2)} \quad . \quad . \quad (125b)$$

$$z^2 = \frac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(c^2-a^2)(c^2-b^2)} \quad . \quad . \quad (125c)$$

By referring to (123) it is easy to see that the quantities (125) are actually all positive. But at the same time we see that, corresponding to a definite system of values of λ, μ, ν, there is not merely one, but eight systems of values of x, y, z; that is, eight different points of space, symmetrically disposed in the eight octants, correspond to each definite system of values of λ, μ, ν. To make the correspondence unique we shall henceforth assume the point to lie in the first octant—that is, we shall assume that x, y, z are all positive.

Above all, we must now prove that the transformation is orthogonal. This is done by calculating the expressions

(111). We therefore differentiate (125a) with respect to λ:

$$2x\frac{\partial x}{\partial\lambda} = \frac{(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)}.$$

Dividing by (125a) we get:

$$\frac{\partial x}{\partial\lambda} = \frac{x}{2(a^2+\lambda)} \text{ and similarly } \frac{\partial x}{\partial\mu} = \frac{x}{2(a^2+\mu)} \quad . \quad . \quad . \quad . \quad (126)$$

If we insert all these values in (111) and then substitute the values for x^2, y^2, z^2 from (125) we see that the conditions for orthogonal transformation are all fulfilled.

We have next to calculate the quantities Λ, M, N. For Λ we get by (112) and (126):

$$\frac{1}{\Lambda^2} = \frac{1}{4}\left\{\frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2}\right\} \quad . \quad (127)$$

To transform this expression conveniently into one which depends only on λ, μ, ν, we first differentiate the general identity (124) with respect to u, obtaining:

$$\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2} = \frac{(u-\lambda)(u-\mu)(u-\nu)}{(a^2+u)(b^2+u)(c^2+u)}$$
$$\cdot\left(\frac{1}{u-\lambda} + \frac{1}{u-\mu} + \frac{1}{u-\nu} - \frac{1}{a^2+u} - \frac{1}{b^2+u} - \frac{1}{c^2+u}\right)$$

and substitute in it $u = \lambda$. It then follows, since only the first term in the outside brackets on the right-hand side remains, that:

$$\frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2} = \frac{(\lambda-\mu)(\lambda-\nu)}{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}$$

Accordingly, by (127):

$$\Lambda = 2\sqrt{\frac{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}{(\lambda-\mu)(\lambda-\nu)}} \quad . \quad . \quad (128a)$$

Likewise:

$$M = 2\sqrt{\frac{(a^2+\mu)(b^2+\mu)(c^2+\mu)}{(\mu-\nu)(\mu-\lambda)}} \quad . \quad (128b)$$

and:

$$N = 2\sqrt{\frac{(a^2+\nu)(b^2+\nu)(c^2+\nu)}{(\nu-\lambda)(\nu-\mu)}} \quad . \quad . \quad (128c)$$

All three are positive real expressions. If we substitute them in (118) we get Laplace's equation in elliptic co-ordinates.

We shall now see whether the equation $\Delta\phi = 0$ can be satisfied by the assumption that ϕ depends only on λ and not on μ and ν, as this cannot be answered without investigation. So we now set $\frac{\partial\phi}{\partial\mu} = 0$ and $\frac{\partial\phi}{\partial\nu} = 0$ and obtain for Laplace's equation by (118), using the expressions (128):

$$\frac{\partial}{\partial\lambda}\left(\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}\,\frac{\partial\phi}{\partial\lambda}\right) = 0 \quad . \quad . \quad (129)$$

from which all the terms containing μ and ν have cancelled out as factors. The particular circumstance that the differential equation (129) contains only the one variable λ shows us that the suggested solution is possible and we can now obtain it easily from (129) by integrating twice. The solution is found to be:

$$\frac{\partial\phi}{\partial\lambda} = \frac{A}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad (A = \text{const.}) \quad . \quad (130)$$

and:

$$\phi = A\int_{\lambda_0}^{\lambda}\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad . \quad . \quad . \quad . \quad (131)$$

By (122) the equipotential surfaces are a family of confocal ellipsoids to which the ellipsoid $\lambda = 0$, that is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad . \quad . \quad . \quad . \quad (132)$$

whose semi-axes are a, b, c, belongs. For $\lambda = \infty$ we obtain the infinitely great spherical surface:

$$\frac{x^2+y^2+z^2}{\lambda} - 1 = 0 \, . \quad . \quad . \quad . \quad (133)$$

whose radius is $\sqrt{\lambda}$. And for $\lambda = -c^2$ we obtain the infinitely thin elliptical disc:

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}\leq 1,\ z=0 \quad . \quad . \quad (134)$$

The two integration constants A and λ_0 in (131) are determined if the values of the potential ϕ on any two equipotential surfaces are given. If, for example, we assume $\phi = 0$ at infinity and $\phi = 1$ on the ellipsoid (132), then $\phi = 0$ for $\lambda = \infty$ and $\phi = 1$ for $\lambda = 0$, and so, by (131), $\lambda_0 = \infty$ and:

$$\phi = l\,.\int_{\lambda}^{\infty}\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad . \quad . \quad (135)$$

where, for brevity, we introduce a positive constant of the dimensions of the reciprocal of a length thus:

$$\int_{0}^{\infty}\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}=\frac{1}{l} \quad . \quad . \quad (136)$$

The expression (135) determines the electric field of the conducting ellipsoid (132), which is charged to unit potential, in the whole of the external space. The density of charge h at any point of the surface follows by (63) from the value of $\left(\frac{\partial\phi}{\partial\lambda}\right)_{\lambda=0}$, but we must observe that the element of length of the normal to the ellipsoid is not $d\lambda$, but $\frac{d\lambda}{A}$. We calculate the total charge e on the ellipsoid most simply not by integrating $hd\sigma$, but by determining the flux of induction through the infinitely distant spherical surface (133) of radius $\sqrt{\lambda}$. For this we have, by (135):

$$\phi = l\,.\int_{\lambda}^{\infty}\frac{d\lambda}{\sqrt{\lambda^3}}=\frac{2l}{\sqrt{\lambda}}.$$

On the other hand, the potential at the great distance $\sqrt{\lambda}$ is, since the ellipsoid can be considered as a single pole in this case:

$$\phi=\frac{e}{\sqrt{\lambda}}.$$

Consequently $e = 2l$. Under present circumstances this is simultaneously the electric capacity C of the ellipsoid, and hence by (136):

$$\frac{1}{C} = \frac{1}{2} \cdot \int_0^\infty \frac{d\lambda}{\sqrt{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}} \quad . \quad . \quad (137)$$

From this it follows, just as in (95), that $C = R$ for a sphere of radius $R = a = b = c$. For $a = b > c$ (flattened ellipsoid of rotation, and for $a > b = c$ (elongated ellipsoid of rotation) the integration of (137) leads to an expression involving circular functions or a logarithmic expression, respectively. In the former we have the special case $c = 0$ (circular disc of radius a):

$$C = \frac{2a}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

In the second case we have for $b = c$, both being infinitely small (infinitely thin wire of length a and diameter $2b$):

$$C = \frac{a}{\log \frac{2a}{b}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

Since the denominator is infinite, C is infinitely small in this case. Hence even if an infinitely thin wire has a finite potential it has only an infinitely small charge and consequently produces only an infinitely weak electric field at finite distances.

§ 26. In this last section of the chapter we shall go a step further in the theory. We have learned two totally different ways of investigating electric fields, which are fully equivalent to each other in electrostatics. Each has its special advantages. The one starts from the theory of contiguous action and links up with the idea of a stream of liquid, the other originates in the theory of action at a distance and operates with Newton's law of gravitation, requiring, however, the introduction of free electricity in place of true electricity. Now we have already explained in § 1 of the present book that there is

only one theory of contiguous action but several theories of action at a distance. We can use this arbitrariness in the theories of action at a distance to reduce all free electricity in principle to true electricity by modifying our point of view to a certain extent. This modification appears here only as a mathematical transformation. Nevertheless, it has proved extraordinarily fruitful in developing the theory—namely in extending it to bodies composed of atoms. For, by (73), we can always write the potential of an electrostatic field, which is due to arbitrary charges situated at the surface-elements $d\sigma$ of conductors and insulators, in the form:

$$\phi = \int \frac{h d\sigma}{r} - \int \frac{h - h'}{r} d\sigma = \phi_0 + \phi_1 \quad . \quad . \quad (140)$$

Here the first integral, ϕ_0 is the potential of the true electricity—that is, the potential which the true charges would call up if the vacuum were the only insulator present; whereas the second integral ϕ_1 which has been taken negative represents the correction which must be applied to ϕ_0 if the electrical properties of the bodies deviate from those of a vacuum. To transform ϕ_1 we introduce the vector:

$$\frac{\boldsymbol{D} - \boldsymbol{E}}{4\pi} = \boldsymbol{M} = \frac{\epsilon - 1}{4\pi} \cdot \boldsymbol{E} \quad . \quad . \quad . \quad (141)$$

In view of (41) and (88) we can then write:

$$\phi_1 = - \int \frac{\boldsymbol{M}'_{\nu'} + \boldsymbol{M}_{\nu}}{r} d\sigma \quad . \quad . \quad . \quad (142)$$

The integration is to be taken over all the bounding surfaces of the conductors and insulators, and the numerator over r denotes the abrupt change which the normal component of the vector $\boldsymbol{M}$ undergoes at the surface-element $d\sigma$; r is the distance of the reference-point x, y, z from the surface-element $d\sigma$, whose coordinates we shall take as ξ, η, ζ. The variables of integration are ξ, η, ζ. But this surface integral may

also be written as a space integral. For if, by integrating by parts according to II (79), we transform the integral:

$$\int\left(M_\xi \frac{\partial \frac{1}{r}}{\partial \xi} + M_\eta \frac{\partial \frac{1}{r}}{\partial \eta} + M_\zeta \frac{\partial \frac{1}{r}}{\partial \zeta}\right) d\tau \quad . \quad . \quad (143)$$

which is to be taken over the whole of infinite space, into another space integral and a surface integral, remembering that on account of the discontinuities of the vector M, the whole space integral is to be treated before the transformation as the sum of the space integrals which are to be taken over the individual homogeneous bodies, just as we had to do in the case of the space integral (37), then the integral (143) comes out as:

$$-\int\left(\frac{\partial M_\xi}{\partial \xi} + \frac{\partial M_\eta}{\partial \eta} + \frac{\partial M_\zeta}{\partial \zeta}\right) \cdot \frac{d\tau}{r} - \int \frac{M'_{\nu'} + M_\nu}{r} d\sigma \quad . \quad (143a)$$

or, since the first integral vanishes on account of (141) and (150), this equals ϕ_1. Hence, substituting, we get:

$$\phi = \int \frac{h d\sigma}{r} + \int\left(M_\xi \frac{\partial \frac{1}{r}}{\partial \xi} + M_\eta \frac{\partial \frac{1}{r}}{\partial \eta} + M_\zeta \frac{\partial \frac{1}{r}}{\partial \zeta}\right) d\tau \quad . \quad (144)$$

According to this expression the potential of the field is composed of the true charges and of the sum of certain potentials which are due to the individual space-elements $d\tau$ of the insulators, since $M = 0$ for conductors. A potential of this kind due to the space-element $d\tau$:

$$\left(M_\xi \frac{\partial \frac{1}{r}}{\partial \xi} + M_\eta \frac{\partial \frac{1}{r}}{\partial \eta} + M_\zeta \frac{\partial \frac{1}{r}}{\partial \zeta}\right) d\tau \quad . \quad . \quad (145)$$

may be interpreted, according to II (339), as the potential of a "double pole" or "dipole," whose moment and axis are represented by the vector $M d\tau$. It is the potential of two infinitely near equally and oppositely charged poles, and is fully analogous to the velocity potential of a double source in hydrodynamics; the moment is the product of

the charge on one pole and the distance; the axis is the direction leading from the negative to the positive pole.

In so far as the volume-element of an insulator is regarded as the seat of an electric dipole, it is said to be electrically or dielectrically "polarized." The vector $\boldsymbol{M}$ of polarization is, by (141), proportional to the local intensity of field $\boldsymbol{E}$:

$$\boldsymbol{M} = \kappa \cdot \boldsymbol{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad (146)$$

where the constant:

$$\kappa = \frac{\epsilon - 1}{4\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (147)$$

is called the electric "susceptibility" of the insulator. Only a pure vacuum is unpolarizable; for all other insulators κ has a positive value, small or great.

The introduction of the electric polarization, which, as we see, renders the conception of free electricity superfluous, has only a formal significance here where we regard the dielectric constant ϵ as given from the outset. But electric polarization acquires a real physical content when we raise the question of the nature and cause of dielectric constants—a question which can, however, be answered only by making atomistic hypotheses—and it becomes imperative to find an answer if we wish to account for those phenomena in which the quantity ϵ no longer acts as a constant. In this case we have to imagine the dielectric effects of an insulator as being due to the actions of an enormous number of small dipoles which lie very close together in an electrically excited vacuum, the moments of the dipoles being determined by the intensity of field, just as in § 16 (Fig. 2) the insulated conductor in the homogeneous field undergoes a definite kind of electric polarization owing to the external intensity of field.

CHAPTER II

ELECTROSTATIC FIELD WITH CONTACT POTENTIALS

§ 27. In the preceding chapter we made use of the simplifying assumption that the contact potential, which is represented by (61) and is constant along the common boundary surface of two substances, but is dependent on their chemical constitution and temperature, can be entirely neglected. Here we shall discuss the characteristic phenomena that appear where contact potentials exist. An important distinction arises according to whether only two substances touch at all the common bounding surfaces between the system of conductors and insulators in question or whether there are also boundary points at which three substances meet simultaneously.

In the former case the occurrence of a contact potential —that is, an abrupt change of potential at the common boundary surface of two substances, no matter how great it may be, has not the slightest influence on the constitution of the electrostatic field. For all the measurable quantities, such as the field strength and the densities of charge of the true and of the free electricity, retain exactly the same meaning as if the potential were continuous throughout, since only the differential coefficients occur in them. The only difference is that the potential ϕ itself undergoes a definite sudden change in the passage from one substance to another. For example, if we have the boundary surface of an insulator and a conductor and call the contact potential $E_{01} = -E_{10}$—this is the change of potential in passing from the insulator (0) to the conductor (1)—the sudden change is, by (61), the same on

the whole boundary surface of the two substances and produces no change in the boundary conditions for the field components and the charges, and hence also no change in the properties of the electrostatic field. For this reason, too, the quantity E_{01} completely eludes measurement.

The position becomes quite different if a substance touches not simply one other substance, but two or more simultaneously. For then there are certain favoured boundary curves in which the three substances meet, and special conditions arise for those singular points. We shall here investigate further the most important case when two conductors 1 and 2 (say copper and zinc) touch each other and are also in contact with a common insulator 0 (say air). In Fig. 5 the conductors—whose form is of no consequence—are conceived as rectangular blocks. Let the singular boundary curve run at right angles to the plane of the drawing, intersecting it at the point P. Let the contact potentials at the three common bounding surfaces be E_{01}, E_{02}, E_{12} with the interpretation which was given above. We can then follow out the behaviour of the potential function ϕ if we take the reference-point along any curve, say, a circle, in the plane of the drawing and encircling the singular point P. We shall begin with a point A in the insulator 0, close to the surface of the conductor 1 and shall call its potential at this point ϕ_1. The potential at a point B beyond the bounding surface is then $\phi_1 + E_{01}$, and this value is retained in the whole interior of the conductor 1 up to the transition from C to D in the conductor 2, where the value changes to $\phi_1 + E_{01} + E_{12}$, which holds for the whole interior of 2. Finally in passing from E to F back into the conductor 0 the potential becomes:

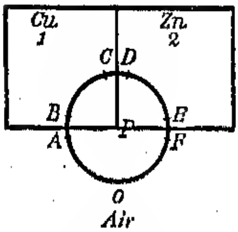

FIG. 5.

$$\phi_1 + E_{01} + E_{12} + E_{20} = \phi_2$$

and this value is constant in the insulator along the whole surface of contact with the conductor 2, while the value ϕ_1 holds everywhere on the surface of 1.

Since in general the sum of the three quantities E will not be equal to zero, the following constant potential difference will exist in the insulator between each two points that lie on the surfaces of the two conductors:

$$\phi_2 - \phi_1 = E_{01} + E_{12} + E_{20} = E \quad . \quad . \quad (148)$$

This is called the *Voltaic contact potential* of the two conductors. Its value is dependent on the chemical constitution of the two conductors and of the insulator, but does not depend on the charge or the potential of the system of conductors. In the case considered (1 copper, 2 zinc, 0 air) E is positive, that is, the potential at the conductor 2 is greater than at the conductor 1. Expressed diagrammatically:

$$E = \text{air} \mid \text{Cu} + \text{Cu} \mid \text{Zn} + \text{Zn} \mid \text{air} > 0 \, . \quad (148a)$$

Since the surfaces of the two conductors represent different equipotential surfaces, lines of force pass through the insulator from the one (2) at the higher potential to the other (1) at the lower potential, say in the form of a circular arc FA. Thus there is an electric field in the insulator whose intensity as we approach the singular boundary curve actually becomes infinite because there is then a finite potential drop over an infinitely short distance.

The problem of determining this electric field presents no obstacles apart from the mathematical difficulties. To overcome them we first suppose that we have obtained the function f_1 which with its first differential coefficients is uniform and continuous in the interior of the insulator; also, it satisfies Laplace's equation, vanishes at infinity and has the value 1 at the bounding surface of the conductor 1 and the value 0 at that of the conductor 2. According to the general theorem proved at the end of § 19, there is only one such function f_1. In the same way we suppose the corresponding function f_2 to have been calculated, which has the value 0 at the boundary

surface of the conductor 1 and the value 1 at that of the conductor 2. The potential at any point of the insulator is then:

$$\phi = \phi_1 f_1 + \phi_2 f_2 \quad . \quad . \quad . \quad . \quad (149)$$

The two constants ϕ_1 and ϕ_2 in this equation can be determined by means of the equation (148) combined with the equation which gives either the total charge on both conductors or the potential of one of them.

The expression for the potential ϕ completely determines the electrostatic field for the insulator in the well-known way, also the densities of charge at the surfaces of the two conductors. This solves the whole problem, since the intensity of field vanishes everywhere in the interior of the conductors. From this we see that the three contact potentials E_{01}, E_{12}, E_{20} occur only in combination in (148), so that it is essentially impossible by means of electrostatic measurements to discover the contribution made by any one of them to their sum E.

A second important consequence is that the electrostatic field does not depend at all on the form of the common boundary surface of the two conductors 1 and 2, since this surface plays no part in calculating the functions f_1 and f_2. If therefore we change the surface, while keeping the singular boundary curve (012) fixed, so that the surface keeps close to the boundary surface (01) along its whole course and so makes the conductor 1 contract into a thin layer, the electric field retains its full intensity. This shews us what an important influence a thin conducting layer over a body has on its electrical properties. If the layer is just thick enough to be regarded as homogeneous it exerts an action electrostatically which is fully equivalent to that of a layer of any thickness whatsoever.

§ 28. Let us now also consider an electrostatic field with contact potentials from the point of view of action at a distance. For simplicity we shall neglect the dielectric properties of the insulator and shall assume $\epsilon = 1$, thus identifying the true electricity with the free elec-

tricity. The question is: how is an electric charge to be distributed over the common boundary surface of two substances so that it produces an abrupt change of the potential at this surface? The answer to this question is obtained if we suppose the abrupt transition of potential first to be replaced by a steep ascent up a short but finite distance and then proceed to the limit. If the potential undergoes a finite change within a very small piece δ of the surface normal, we have, by § 16, the electrical field of a condenser, and the change in potential, which we shall call $\phi' - \phi$, is, by (69), connected with the density of charge h of the condenser by the relation:

$$\phi' - \phi = 4\pi h \cdot \delta \quad . \quad . \quad . \quad . \quad . \quad (150)$$

If we pass to the limit $\delta = 0$, then, if the sudden potential change is to remain constant, the product $h \cdot \delta$ must have a finite value g; that is, at every surface-element $d\sigma$ we have an electric dipole whose moment is the product of the charge $hd\sigma$ and the distance δ between the two charges, that is, $gd\sigma$, and whose axis is the direction of the surface normal ν, taken from the negative to the positive charge. A charge of this kind arranged in the form of electric dipoles on a surface is called an electric *double-layer* of moment g (>0). As the reference-point passes from the negative to the positive side this double-layer causes an abrupt change of potential:

$$\phi' - \phi = 4\pi g. \quad . \quad . \quad . \quad . \quad . \quad (151)$$

Since this change is constant all along the surface we must also regard g as constant and so consider the double-layer as homogeneous.

Following on this we shall now establish generally the potential function of a homogeneous double-layer spread over a given arbitrary surface and having the moment g and the direction of axis ν. For this we get by comparison with (145) the expression:

$$\phi = \int \left(g_\xi \frac{\partial \frac{1}{r}}{\partial \xi} + g_\eta \frac{\partial \frac{1}{r}}{\partial \eta} + g_\zeta \frac{\partial \frac{1}{r}}{\partial \zeta} \right) d\sigma$$

where $d\sigma$ denotes an element of surface, $\xi\ \eta\ \zeta$ its co-ordinates, and $g_\xi\ g_\eta\ g_\zeta$ the components of the vector g which lies in the direction ν. Or also:

$$\phi = \int g \frac{\partial \frac{1}{r}}{\partial \nu} d\sigma \quad . \quad . \quad . \quad . \quad . \quad (152)$$

and since the double-layer is homogeneous:

$$\phi = g \int \frac{\partial \frac{1}{r}}{\partial \nu} d\sigma \quad . \quad . \quad . \quad . \quad . \quad (153)$$

This quantity allows of a simple geometrical interpretation. For if we write:

$$\phi = -g \int \frac{1}{r^2} \frac{\partial r}{\partial \nu} d\sigma \quad . \quad . \quad . \quad . \quad . \quad (154)$$

and reflect that $\frac{\partial r}{\partial \nu}$ is the cosine of the angle which the radius vector r, drawn from the reference-point (xyz) to the surface-element $d\sigma$, makes with the normal to the surface ν (this angle is positive if r increases simultaneously with ν, that is when the reference-point lies on the negative side of the double-layer), then the product of this cosine with $d\sigma$ is the normal cross-section of the infinitely thin cone which has its vertex in the reference-point and which is cut out of the double-layer by the surface-element $d\sigma$. This cross-section divided by r^2 gives the small area which the same cone cuts out of the unit sphere described about the point of reference as centre, that is, the "angle of aperture" ("solid angle" or "apparent size") $d\Omega$ of the surface-element $d\sigma$ from the point of view of the reference-point. Hence by (154) we get on integrating that:

$$\phi = \pm g \cdot \Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

where Ω denotes the absolute value of the angle of aperture of the whole double-layer and the positive or negative sign is to be taken according as the double-layer turns its positive or its negative side to the reference-point.

The most remarkable feature about the preceding relation is that the angle of aperture Ω and consequently the potential ϕ depend only on the boundary curve of the double-layer and not in the least on the form of the surface over which it is distributed. For if we alter the surface, while keeping the boundary fixed, the angle of aperture obviously always remains the same.

If the double-layer has no boundary at all, that is, if it completely encloses a certain volume of space, its potential ϕ can easily be given, since for any arbitrary reference-point that lies within the space Ω is equal to the whole surface of the unit sphere, that is, 4π. Consequently, if the inner side of the double-layer is positive, $\phi = 4\pi g$. But for an external point the double-layer resolves into the two parts marked off by the cone of contact from the reference-point to the surface. The angles of aperture of these two parts of the surface are equal, but their charges are opposite inasmuch as the one turns its positive side to the reference-point and the other its negative side. Hence the potentials due to them cancel out and $\phi = 0$. Hence when the reference-point passes through the double-layer the potential changes suddenly by the amount $4\pi g$, which corresponds exactly with the equation (151) which was deduced for another case.

It can easily be seen that the equation gives quite generally the relationship between the sudden change of potential and the moment of the double-layer. For the potential "jump" can depend only on the nature of the double-layer at the point where the reference-point passes through it, but not on its nature in surface-elements at finite distances, whose potentials are all continuous.

We next inquire into the behaviour of the first differential coefficients of a homogeneous double-layer during the passage of the reference-point through the layer. It is obvious that the tangential differential coefficients remain continuous, since we get the value zero for the difference of the differential coefficients of ϕ and ϕ' with respect to any direction lying on the surface, when we

differentiate the equation (151), which holds all along the surface. But neither do the normal differential coefficients undergo a sudden change. For in the case of a closed double-layer these are, by what has been said above, equal to zero both in the outward and in the inward direction, that is, they are continuous; and they must retain the same property for *any* double-layer, since a boundary curve that lies at a finite distance can have no influence on the properties of continuity.

The first differential coefficients of the potential function ϕ also give the direction and magnitude of the force exerted by the double-layer on a charged reference-point. The force is zero throughout for a closed double-layer, both inside and outside, and for an unclosed double-layer the force depends only on the boundary curve. In other respects the position of the surface for the electric field produced by the double-layer has no physical significance at all. The electric lines of force start from the positive layer (§ 15) and pass outside around the boundary curve to the negative layer. The beginning and the end of a line of force coincide, however, in such a way that the beginning of the line of force can be regarded simply as the continuation of the end, without a physical discontinuity manifesting itself. In the neighbourhood of the boundary curve the lines of force that run round it contract to small circles, in which the field strength is very great, because the finite drop of potential occurs over a short distance. The conditions here are precisely like those in the case of the velocity potential and the stream-lines of an infinitely thin vortex-ring in an incompressible liquid. Cf. II, § 75.

For example, let us calculate the potential and the field strength of a *circular* double-layer of radius R at a reference-point on its axis. We take the axis of the double-layer as our positive z-axis. The solid angle subtended by a reference-point on the positive side of the layer at a distance z from the centre of the disc is then:

$$\Omega = \iint \sin\theta \, . \, d\theta \, . \, d\phi$$

where the limits for ϕ are 0 and 2π and for θ are 0 and $\tan^{-1}\frac{R}{z}$. This gives us:

$$\Omega = 2\pi\left(1 - \frac{z}{\sqrt{R^2 + z^2}}\right) \quad . \quad . \quad . \quad . \quad (156)$$

and so by (155):

$$\phi = 2\pi g\left(1 - \frac{z}{\sqrt{R^2 + z^2}}\right) \quad . \quad . \quad . \quad . \quad (157)$$

whereas for a reference-point on the negative side of the layer ($z<0$) the following relation holds:

$$\phi = -2\pi g\left(1 + \frac{z}{\sqrt{R^2 + z^2}}\right) \quad . \quad . \quad . \quad . \quad (157a)$$

If the reference-point passes through the layer from the negative to the positive side the potential jumps from $-2\pi g$ to $+2\pi g$, corresponding to equation (151). On the other hand, the differential coefficient:

$$\frac{\partial\phi}{\partial z} = -\frac{2\pi R^2 g}{(R^2 + z^2)^{\frac{3}{2}}} = -E_z \quad . \quad . \quad . \quad . \quad (158)$$

is continuous throughout, even in passing through the layer. It retains its value if we suppose the double-layer to be distributed over any surface whatsoever bounded by the circle in question. Only the circle itself has a physical significance for the field excited by the double-layer. The field-strength E_z is, as we see, positive throughout and symmetrical with respect to the plane of the circle. On the positive side the repulsion of the positive layer predominates, on the negative side the attraction of the negative layer predominates. At infinity the field-intensity is equal to zero; it attains its maximum in the plane of the circle, namely:

$$E_z = \frac{2\pi g}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

§ 29. We have now become acquainted with three different kinds of potential functions in all: those relating to masses in space, surface masses and homogeneous

double-layers. Poisson's equation I (132) for the space-density k is characteristic of the first of these; in this case the potential and its first differential coefficients remain continuous. The second kind is characterized by the discontinuity (85) in the normal differential coefficient due to the surface-density h; the potential again remains continuous and Laplace's equation is everywhere satisfied. The third kind is characterized by the discontinuity (151) of the potential itself caused by the moment g; here the differential coefficients remain continuous—and Laplace's equation is again satisfied.

Conversely, every arbitrary given uniform function f of the space co-ordinates x, y, z which with its differential coefficients can also be discontinuous at definite surfaces and which vanishes at infinity, can always be represented in a unique way as a potential function of gravitating masses situated in finite regions, the masses being distributed partly in space, partly over surfaces and partly as double-layers. For if we use (151) to calculate from the abrupt transition of f at the surfaces of discontinuity the value of g and (85) to calculate the value of h from the sudden change in the normal differential coefficient $\frac{\partial f}{\partial \nu}$, and finally the value of k from the value of Δf by means of Poisson's equation (I (132)), and then write down the potential function:

$$\phi = \int g \frac{\partial \frac{1}{r}}{\partial \nu} d\sigma + \int \frac{h d\sigma}{r} + \int \frac{k d\tau}{r} \quad . \quad . \quad . \quad (160)$$

we find that the difference $\phi - f$, regarded as a function of x, y, z, has the following properties. It is uniform and, with its first differential coefficients, everywhere continuous, since the sudden changes in ϕ and f under consideration everywhere exactly cancel. Further, it satisfies Laplace's equation and vanishes at infinity. Hence by the general theorem proved in § 19 it is equal to zero, that is, $\phi = f$.

The three characteristic relations between the space-density k, the surface-density h and the density g of the double-layer, on the one hand, and the potential function ϕ, on the other, are closely connected with each other and may also be formally derived from one another if we replace every discontinuity by a finite even if rapid change and subsequently proceed to the limit. For example, if in the case of a plane, uniformly charged conducting surface whose normal is ν we suppose the geometrical surface to be replaced by a spatial non-homogeneous layer of very small thickness δ, then Poisson's equation:

$$\operatorname{div} \phi = \frac{\partial^2 \phi}{\partial \nu^2} = -4\pi k$$

holds for the spatial divergence of ϕ. By integrating this equation from $\nu = 0$ to $\nu = \delta$ we get:

$$\left(\frac{\partial \phi}{\partial \nu}\right)' - \left(\frac{\partial \phi}{\partial \nu}\right) = -4\pi \int_0^\delta k d\nu = -4\pi h$$

which is identical with the condition (85) for the surface divergence.

In the same way we can imagine a sudden change in the value of the potential function ϕ to be replaced by a steep but finite rise within the layer δ. We then see directly from the course of the curve for ϕ (Fig. 6) that its curvature as represented by the second differential coefficient $\frac{\partial^2 \phi}{\partial \nu^2}$ has very great values at the beginning and at the end of the transition layer. These values are, however, of opposite sign, corresponding to the two opposite charges at the ends of the double-layer.

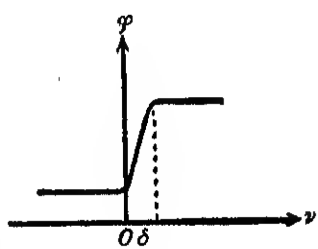

FIG. 6.

On account of its uniformity this method of representation is very convenient for certain rather general

investigations, since it avoids all kinds of discontinuities and hence attains its object with only the spatially distributed density of charge k and Poisson's equation. On account of the finite transition layers, however, the bodies can then no longer be assumed to be homogeneous throughout.

§ 30. Reverting again to the arguments put forward at the beginning of this chapter, the theorems there deduced now appear to us in a new light, when regarded from the point of view of action at a distance. For example, if in a system of conductors and insulators only two substances at a time touch at any of the common surfaces that occur, it is immediately clear that the occurrence of contact potentials can have no influence on the electrostatic field. For the double-layers which correspond to the abrupt changes of potential are all closed, and hence exert no physical action either inside or outside.

But if two conductors 1 and 2 are in contact with each other and with an insulator 0, as in Fig. 5, then we have three homogeneous double-layers (01), (12), (20) with a common boundary curve (012); and since the action of a homogeneous double-layer depends only on the boundary curve, we can imagine all three layers to be distributed over one and the same surface with one and the same boundary curve. Hence it follows immediately that only the sum E, given by (148), of the three corresponding contact potentials, namely the Voltaic contact potential, comes into account for the electrostatic field in the insulator, there being no possibility of separating the individual summands from each other by means of electrostatic measurements.

If in place of only the conductors 1 and 2 we have a greater number of conductors 1, 2, . . . n connected with one another in succession by conducting wires the system constitutes a so-called *Voltaic chain* or series. In place of (149) we then have for the potential ϕ of the electric field in the insulator the more general equation :

$$\phi = \phi_1 f_1 + \phi_2 f_2 + \phi_3 f_3 + \ . \ . \ . \ \phi_n f_n \quad . \quad (160a)$$

where the functions $f_1, f_2, f_3 \ldots f_n$ are equal to one at the surface of the conductor carrying the same suffix, but are equal to zero at the surfaces of all the other conductors, whereas the constants $\phi_1, \phi_2, \phi_3 \ldots \phi_n$, which give the values of the potential in the insulator in the immediate neighbourhood of the surface of the conductor in question, are uniquely determined by the Voltaic contact potentials (148) taken in conjunction either with the equation which gives the total charge of the system of conductors or with that which gives the potential in one of the conductors.

The potential of the whole chain or the potential difference $\phi_n - \phi_1$, is then obtained directly by making the reference-point pass out of the insulator into the conductor 1 and through all the conductors in turn back into the insulator thus:

$$\phi_n - \phi_1 = E_{01} + E_{12} + E_{23} + \ldots E_{n-1,n} + E_{n,o} = E. \quad (161)$$

A particularly important case is that in which the substance of the last conductor n is of the same constitution as that of the first conductor 1. For then we clearly have $E_{01} + E_{n_o} = E_{01} + E_{10} = 0$, and so:

$$E = \phi_n - \phi_1 = E_{12} + E_{23} + \ldots E_{n-1,n}. \quad (162)$$

thus the potential of the chain is quite independent of the nature of the insulator. This kind of potential is distinguished by being called the *galvanic potential.* If, for example, the conductors 1, 2, 3, 4, 5 are, in order, copper, zinc, solution of zinc sulphate, solution of copper sulphate, copper, then the galvanic potential of this chain, the so-called Daniell cell, is:

$$E = \mathrm{Cu}\,|\,\mathrm{Zn} + \mathrm{Zn}\,|\,\mathrm{Zn\,SO_4\,aq} + \mathrm{Zn\,SO_4\,aq}\,|\,\mathrm{Cu\,SO_4\,aq} + \mathrm{Cu\,SO_4\,aq}\,|\,\mathrm{Cu}. \quad (163)$$

In contrast with the case of the Voltaic-potential the insulator plays no part here.

§ 31. A totally new case arises if the last member of the chain, the conductor n, is connected by means of a conducting wire, say, with the first member of the chain,

the conductor 1; that is, instead of the "open" chain so far discussed we now use a "closed" chain. For then the system of conductors forms a doubly-connected space (cf. II, § 69), that is, not all the curves which run from one conductor to another in the conducting space can be transformed continuously into one another. In general, therefore, in this case we arrive at two different values for the potential difference of any two conductors, say 1 and 2, according to the way in which the reference-point leads from one conductor to the other. For example, the potential difference between 1 and 2 is equal to E_{12}, but it is also equal to $E_{1n} + E_{n,\,n-1}, + \ldots E_{43} + E_{32}$, and since electric potential is by its very nature one-valued, a contradiction manifests itself in general, that is, an electrostatic field is not possible at all in the case under consideration. It is only in the special circumstances when the two conditions mentioned are equal, or when, written symmetrically :

$$E_{12} + E_{23} + \ \ldots \ + E_{n-1,\,n} + E_{n_1} = 0 \quad . \quad (164)$$

that the conditions necessary for the electrostatic field can be fulfilled. The relation expressed in (164), which merely refers to the chemical constitution of the conductors, is called the *Law of Potential Sequence*. Thus an electrostatic field is possible in a closed series of conductors only if the conductors obey the law of potential sequence. Hence we shall assume the equation (164) as valid. For the present we must exclude the contrary case, to which we shall revert later in Chapter V.

Conductors which obey the law of potential sequence are called "conductors of the first class," the remainder are called "conductors of the second class." If we have three conductors 1, 2, 3 of the first class then, by (164), the following relation holds for them :

$$E_{12} + E_{23} + E_{31} = 0$$

or :

$$E_{12} = E_{32} - E_{31}.$$

In this equation the contact potential of the conductors

1 and 2 appears as the difference of two quantities, of which the first does not depend at all on the conductor 1 and the second does not depend on the conductor 2. Since the conductor 3 is otherwise quite arbitrary and its influence cancels out in the two terms of the difference, it enters into the quantities E_{32} and E_{31} only in the form of an arbitrary additive constant. If we suppress this constant altogether in our symbols we may write the last equation simply as:

$$E_{12} = E_2 - E_1 \quad . \quad . \quad . \quad . \quad (165)$$

where we leave an additive constant undetermined in the quantities E_1 and E_2. This constant can clearly always be chosen so that the two quantities E_1 and E_2 are positive.

The relation (165) gives the necessary and sufficient condition that two conductors 1 and 2 should be of the first class. We see at once, in fact, that in every chain of conductors which satisfy the equation (165) the law of potential sequence (164) is obeyed identically.

The equation (165) enables us to arrange in a series all conductors of the first class, namely, in the order of the numerical values of the positive quantities E_1, E_2, E_3 . . ., starting from the smallest and ending with the greatest value: the galvanic sequence. From this sequence we can immediately obtain the sign of the contact potential of any two conductors. For by (165) every conductor of the series has, when touched by a conductor which precedes it in the series, a higher potential than the latter and vice versa.

By (148) and (165) the following relation holds for the voltaic potential between two conductors of the first class:

$$\phi_2 - \phi_1 = E_{01} + E_2 - E_1 + E_{20}$$

or:

$$\phi_2 - \phi_1 = E'_2 - E'_1 \quad . \quad . \quad . \quad . \quad (166)$$

where we have set

$$E'_1 = E_1 + E_{10},\ E'_2 = E_2 + E_{20} \quad . \quad . \quad (167)$$

From this we see that there is a potential sequence not only for the galvanic contact potentials, but also for the voltaic contact potentials between conductors of the first class. But the sequence of the conductors is in general different in these two cases; for in the case of the voltaic potentials it is essentially dependent on the chemical constitution of the insulator 0. Disregard of this circumstance has given rise to a number of controversies which occupy a large space in the literature of the subject.

§ 32. Let us finally discuss briefly the question of the chemical difference between conductors of the first and the second class. We consider a closed chain in which one or more conductors of the second class have been inter-linked. Under no circumstances is an electrostatic field possible in such a chain; hence an electric current will manifest itself. But it is impossible for the system of conductors to remain completely unchanged with this current passing. For otherwise it would last for ever and the effects which it produces would continue unceasingly to arise from nothing, which would contradict the universal principle of conservation of energy. From this we must conclude, then, that the current must effect a certain change in the contact potentials and hence in the chemical constitution of the surfaces of the conductors, namely in those conductors which give rise to it, that is, the conductors of the second class in the chain. If we call a conductor whose chemical nature is changed when a current flows through it an electrolyte, then we can enunciate the following general theorem which has been confirmed in the widest sense experimentally: all conductors of the second class are electrolytes. On the other hand all conductors that are not chemically changed by an electric current, that is, metals, are necessarily conductors of the first class.

CHAPTER III

THE MAGNETOSTATIC FIELD

§ 33. Reverting to the general results derived in § 13, we shall now consider the laws of magnetostatics, which we shall treat on exactly the same lines in principle as those used in electrostatics. The laws of a magnetostatic field in a system of homogeneous bodies are expressed in the following equations which hold in the interior of a body: (51) div $\boldsymbol{B} = 0$ and (57) curl $\boldsymbol{H} = 0$, which in combination with (30) gives $\boldsymbol{B} = \mu\boldsymbol{H}$; and for the common surface between two bodies the equation (52): $\boldsymbol{B}'_{\nu'} + \boldsymbol{B}_\nu = 0$.

These equations resemble exactly the corresponding equations for electrostatics. But they are simpler in that the magnetic boundary condition (52) does not, as in the corresponding electrical boundary condition (41), contain the quantity h which we called the true density of charge in § 20. In other words, in magnetism the true charge is quite generally equal to zero, not only the volume charge (volume divergence of the induction), but also the space charge (space divergence of the induction).

Hence when we break a magnet each piece again forms a complete magnet. It is not possible to separate magnetism into positive and negative in space in a manner corresponding to the separation of electricity into positive and negative; nor is there any process in magnetism corresponding to conduction by an electric current. Magnetically all substances are insulators.

Although in the light of these remarks magnetic fields show a much simpler aspect than electric fields, yet in another no less important respect they show themselves

to be considerably more complicated. For rather accurate measurements have shown that in a fair number of widely distributed substances, known as "ferromagnetic," which include iron, cobalt, nickel, and manganese-alloys, the magnetic state does not depend alone on the momentary intensity of field, as we assumed in § 3, or that, as we may say in the language of II, § 21, the substances mentioned do not behave perfectly elastically in magnetic respects. For if such a body is situated in a magnetic field whose intensity alternately increases and decreases, the magnetic induction in it has not always a definite value for a definite intensity of field, but is smaller when the intensity of field is increasing than when it is decreasing; in a certain sense it lags behind for field-strength. Hence we can no longer talk of the equation (30) being valid in this connection. This phenomenon is called "hysteresis." As it would require much space to consider hysteresis and its implications here, we have excluded it from our present discussion.

But we must also call attention to another phenomenon of fundamental importance which likewise causes a deviation from the equation (30). In the case of ferromagnetic substances the induction, even if it depends only on the momentary field-strength, is in general not in the same direction as, nor proportional to, the field-strength; this means that instead of the vector equation (30) there is a totally different relation between $\boldsymbol{H}$ and $\boldsymbol{B}$. Magnetism also exhibits a more complicated behaviour than electricity in the fact that whereas the dielectric constant ϵ is essentially dependent on the field-strength and is always greater than 1, the magnetic permeability μ, even when it is constant for a substance, can be both greater and less than 1; in the former case the substance is said to be "paramagnetic" (platinum, oxygen, nitrogen), in the latter case "diamagnetic" (bismuth, copper, water, hydrogen).

In view of these circumstances, it is desirable to discuss the properties of a magnetostatic field without taking

equation (30) into account at all at first, since this is the only equation which involves the specific behaviour of bodies.

§ 34. Corresponding to the equation (42) which holds for the true electric charge, the following theorem follows from (51) and (52) quite generally for the induction $\boldsymbol{B}$: *the total flux of magnetic induction through the surface of any volume of space vanishes*, or

$$\oint \boldsymbol{B}_\nu \, d\sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad (168)$$

where the ring on the integral sign is to signify that we are dealing with a closed surface. For an *unclosed surface* the flux of magnetic induction consequently *depends only on the boundary curve*, and not on the shape of the surface. This can be readily deduced by reflecting that two different surfaces having a common boundary curve form a closed surface for which the equation (168) is then valid. Accordingly, in a magnetic tube of induction (§ 15) the flux of induction everywhere has the same value and passes unchanged through every surface of separation between two bodies.

On the other hand, we see from (57) that a potential ϕ exists for the field-strength $\boldsymbol{H}$.

$$\boldsymbol{H} = -\operatorname{grad} \phi . \quad . \quad . \quad . \quad . \quad (169)$$

We assume the magnetic potential ϕ to be continuous throughout, even at the common surface of two different substances, since, in contrast with electric potential, there is not a single fact to compel us to assume discontinuity. Moreover, the magnetic potential is uniform, since the equation (169) holds in the whole magnetostatic field, which forms a singly connected space.

Accordingly, then, we must distinguish carefully between the properties of the field-strength $\boldsymbol{H}$ and those of the magnetic induction $\boldsymbol{B}$, between lines of force and lines of induction, between flux of force and flux of induction. In ferromagnetic substances only the lines of force are normal to the system of equipotential surfaces $\phi = \text{const.}$ and run in the direction of decreasing

potential, whereas the lines of induction can run in quite another direction, even in the opposite direction. On the other hand, the simple equation (168) holds only for the flux of induction. The flux of force through a closed surface can very well differ from zero; for example, free charges, both space and surface charges, can occur, at which lines of forces arise or vanish. We shall meet with some simple examples of these conditions below in §§ 37 and 38.

If, by I (132) we set:

$$\frac{\operatorname{div} \boldsymbol{H}}{4\pi} = -\frac{1}{4\pi} \cdot \Delta\phi = k' \quad . \quad . \quad . \quad (170)$$

(free space-density of charge)

and, by (85):

$$\frac{\boldsymbol{H}_\nu + \boldsymbol{H}'_{\nu'}}{4\pi} = -\frac{1}{4\pi}\left(\frac{\partial\phi}{\partial\nu} + \frac{\partial\phi'}{\partial\nu'}\right) = h' \quad . \quad . \quad (171)$$

(free surface-density of charge)

hence by § 29:

$$\phi = \int\frac{h'd\sigma}{r} + \int\frac{k'd\tau}{r} \quad . \quad . \quad . \quad . \quad . \quad (172)$$

According to this we can always picture the magnetic field strength by means of the attractive and repulsive forces of the free charges. If we integrate the equation (170) over the whole of infinite space and the equation (171) over all the surfaces of separation of the bodies two at a time, the sum of these two integrals is, by II (82), identically equal to zero, that is, the total free charge of a magnetic field is always equal to zero.

Just as the magnetic potential can be represented by the free charge, so it can be expressed in terms of a polarization of the volume-elements of all the bodies, in exactly the same way as was done in § 26 for the electric potential. For if, as in (141), we set:

$$\frac{\boldsymbol{B} - \boldsymbol{H}}{4\pi} = \boldsymbol{M} \quad . \quad . \quad . \quad . \quad . \quad . \quad (173)$$

and consider the expression (143a), then, on account of the relations (51), (52), (170), (171) this expression comes out perfectly identical with the magnetic potential (172). Thus, if we substitute (143) for (143a) we get:

$$\phi = \int \left(M_\xi \frac{\partial \frac{1}{r}}{\partial \xi} + M_\eta \frac{\partial \frac{1}{r}}{\partial \eta} + M_\zeta \frac{\partial \frac{1}{r}}{\partial \zeta} \right) d\tau \quad . \quad . \quad (174)$$

The vector $\boldsymbol{M} \,.\, d\tau$ is the "magnetic polarization" or the "magnetization" of the volume element $d\tau$; its absolute value is the "magnetic moment." The volume element forms a magnetic dipole whose axis points from the negative to the positive pole and whose moment is given by the product of the charge at either pole and the distance between the poles.

§ 35. Whereas the theorems in the preceding paragraph hold for all bodies without exception, we shall now consider the special cases due to the particular behaviour of certain substances. In each case we deal either with equation (30) or with what has to take the place of this equation, namely the characteristic relation between the induction $\boldsymbol{B}$ and the field-strength $\boldsymbol{H}$ of the substance in question, or between the magnetization $\boldsymbol{M}$ defined by (173) and the field-strength $\boldsymbol{H}$, since this can be formulated more conveniently.

In *paramagnetic* and *diamagnetic* substances for which the equation (30) holds and μ is constant, we have:

$$\boldsymbol{M} = \kappa \boldsymbol{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad (175)$$

where:

$$\kappa = \frac{\mu - 1}{4\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (176)$$

The "magnetic susceptibility" κ of the substance is positive for paramagnetic, negative for diamagnetic substances and zero in a vacuum. Thus in paramagnetic substances the axis of magnetization, that is, the direction from the negative to the positive pole of the excited dipole, coincides with the direction of the field-strength,

whereas in diamagnetic substances it points in the opposite direction.

Ferromagnetic substances behave quite differently. In the first place, the magnetization in their case does not vanish with the field-strength; rather, when $\boldsymbol{H} = 0$, $\boldsymbol{M}$ in general retains a finite value which is also called "permanent magnetism." The ideal limiting case is given by the so-called "rigid magnets," for which the magnetization:

$$\boldsymbol{M} = \boldsymbol{M}_p \quad . \quad . \quad . \quad . \quad . \quad . \quad (177)$$

is absolutely constant and quite independent of the field-strength. A more general type of magnetic behaviour is obtained by combining (175) and (177):

$$\boldsymbol{M} = \boldsymbol{M}_p + \kappa \boldsymbol{H} \quad . \quad . \quad . \quad . \quad . \quad (178)$$

or, by (173):

$$\boldsymbol{B} = 4\pi \boldsymbol{M}_p + \mu \boldsymbol{H} \quad . \quad . \quad . \quad . \quad . \quad (178a)$$

where the second term on the right-hand side is called, in contrast with the first term, the "temporary magnetism" or "temporary induction," respectively. Of course the vectors $\boldsymbol{M}_p$ and $\boldsymbol{H}$ in general have different directions. In many ferromagnetic substances when $\boldsymbol{M}_p$ becomes vanishingly small the susceptibility κ assumes enormous values. Such substances are said to be magnetically "soft." Strictly speaking, however, there is no substance for which κ is absolutely constant. For when the field-intensity $\boldsymbol{H}$ is increased without limit, $\boldsymbol{M}$ never increases without limit, but approaches a definite limiting value which is characteristic of the substance and is called the "saturation limit" of the magnetization. To give due consideration to this fact we must therefore assume that the susceptibility κ in (178) vanishes when the field-strength becomes infinitely great.

As soon as we know the magnetic equation of state characteristic for every body, say in the form of (175), (177), (178) or (178a), we can uniquely determine the properties of the magnetostatic field with the help of the laws of § 34.

A moment's consideration shows that to maintain a finite magnetostatic field permanent magnetism is necessary. For if only paramagnetic and diamagnetic substances are present all the field equations are satisfied automatically, since both $\boldsymbol{H}$ and $\boldsymbol{B}$ vanish everywhere.

Further, we note for later use that the expression (4) for the magnetic energy is valid also for permanent magnets of constant permeability μ, of the type (178a); this is immediately apparent if we start from the relation (32a), which is generally valid, and consider that in the case in question $d\boldsymbol{B} = \mu \, . \, d\boldsymbol{H}$.

§ 36. To show how the theorems which have been derived are applied, we begin with the simplest case of a rigid magnet and consider first a *rigid linear magnet*, that is, a magnetic needle *in vacuo* of arbitrary length and curved in any arbitrary way. Let the infinitely small cross-section of the needle be f, its length l, the element of length $d\lambda$, the magnetization $\boldsymbol{M}\lambda$, which is any arbitrary given function of λ, and let $\boldsymbol{M}_\lambda$ be in the same direction as λ.

Then by (174) the potential of the corresponding magnetic field is, since the magnetization is zero in a vacuum:

$$\phi = \int_0^l \boldsymbol{M}_\lambda \frac{\partial \frac{1}{r}}{\partial \lambda} \cdot d\tau = \int_0 f\boldsymbol{M}_\lambda \cdot \frac{\partial \frac{1}{r}}{\partial \lambda} d\lambda \quad . \quad . \quad (179)$$

or, if we set $f\boldsymbol{M}_\lambda = M$:

$$\phi = \int_0 M \frac{\partial \frac{1}{r}}{\partial \lambda} d\lambda.$$

Integrating by parts:

$$\phi = \frac{M_1}{r_1} - \frac{M_0}{r_0} - \int_0^l \frac{\partial M}{\partial \lambda} \cdot \frac{d\lambda}{r} \quad . \quad . \quad . \quad (180)$$

where the indices 0 and 1 refer to the two end-points of the needle. According to this the magnetic needle acts outwardly exactly like two magnetic poles at the ends of

the needle, carrying the charges M_1 and $-M_0$, together with a charge of density $-\frac{\partial M}{\partial \lambda}$ distributed continuously over the whole length of the needle. The sum of all these free charges is, of course, equal to zero, as it must be by § 34. If M is constant along the whole needle (uniform magnetization) then $M_1 = M_0$, and:

$$\phi = \mathrm{M}_0\left(\frac{1}{r_1} - \frac{1}{r_0}\right) \quad . \quad . \quad . \quad . \quad . \quad (181)$$

In this special case the needle can be replaced by the two end-poles alone.

If the two poles coincide, that is, if the needle forms a closed curve, a so-called ring-magnet, $\phi = 0$ and the magnetic field vanishes entirely, from which the theorem follows that a uniformly magnetized closed magnet of any shape produces no external magnetic effect.

If the needle is so long that one of the two poles is very far from the reference-point, only the action of the other pole remains. Hence a single positively or negatively charged magnetic pole can be realized in nature to any degree of approximation by a uniformly magnetized magnetic needle of appropriate length.

The rectilinear magnets used in practice are never uniformly magnetized; rather, M, assumed positive, is greatest in the centre and decreases symmetrically towards both ends. Hence $\frac{\partial M}{\partial \lambda}$ is equal to zero in the centre, is positive for $\lambda < \frac{l}{2}$, and equally negative for $\lambda > \frac{l}{2}$. So the continuously distributed free density of charge has the same sign on each side as the charge of the corresponding end-pole. Since its amount is equally great on both sides, the two poles are also equally charged. The charges which have yet to be added to the pole-charges have the same effect for far-distant reference-points as if the two poles were slightly displaced from the ends of the needle towards the middle, so that the distance

between the poles amounts to about $\frac{5}{6}$ of the length of the needle.

§ 37. If we now take a rigid magnet of arbitrary size in a vacuum, but of *uniform magnetization* $\boldsymbol{M}$, whose axis we take as our z-axis, then by (174) we have inside and outside:

$$\phi = \boldsymbol{M}\int \frac{\partial \frac{1}{r}}{\partial \zeta} d\tau \quad . \quad . \quad . \quad . \quad . \quad (182)$$

where the integration is to be taken over the volume of the magnet.

Or:

$$\phi = - \boldsymbol{M}\int \frac{\cos (\nu z)}{r} d\sigma. \quad . \quad . \quad . \quad . \quad (183)$$

According to this the magnet acts inside and outside its own volume like a magnetic charge distributed with the density $- \boldsymbol{M} \cos (\nu z)$ over its surface. The amount of this density of charge can easily be pictured in the following way. Let us suppose the magnet displaced to an infinitely small extent in the direction of the positive z-axis; then each element $d\sigma$ of its surface sweeps out an infinitely small volume of the form of an oblique cylinder of base $d\sigma$. The volume of this cylinder is, if we disregard the sign, proportional to $\cos (\nu z) d\sigma$ and so corresponds exactly to the amount of the free charge on $d\sigma$. On the front surface of the displaced magnet the charge is positive, on the rear surface it is negative, while the neutral zone (§ 16) occurs at the points where the surface runs parallel to the z-axis.

The lines of force pass from the positive to the negative charge both inside the magnet as well as outside through the vacuum, so that each positively charged surface-element sends out a tube of force in both directions. On the other hand, by (34) the tubes of induction pass continuously and with a constant flux of induction through the surface of the magnet, and since they coincide in the vacuum with the tubes of force, inside the magnet they

run in the opposite direction to the tubes of force, that is, from the negative to the positive charge. Hence the lines of induction and tubes of induction are closed configurations which, however, are bent at the surface owing to the discontinuity in the tangential components of the induction.

To calculate the potential ϕ in the present case it is better not to use the equation (183), but rather the following form of (182), corresponding to the transformation of II (339) into II (340):

$$\phi = -M\int \frac{\partial \frac{1}{r}}{\partial z}\, d\tau$$

or:

$$\phi = -M\frac{\partial \psi}{\partial z} \qquad \qquad (184)$$

where:

$$\psi = \int \frac{d\tau}{r} \qquad \qquad (185)$$

signifies the Newtonian potential function of the volume of the magnet imagined filled with mass of density 1.

Let us perform the calculation for the case of a rigid uniformly magnetized sphere of radius R situated in a vacuum. By (185) and I, § 38 we have for an external reference-point at the distance $r_0 (> R)$ from the centre:

$$\psi_a = \frac{4}{3}R^3\pi \cdot \frac{1}{r_0}$$

but for an internal point ($r_0 < R$):

$$\psi_i = \frac{2\pi}{3}(3R^2 - r_0^2).$$

From this it follows by (184), since $r_0^2 = x^2 + y^2 + z^2$, that the magnetic potential in outside space is:

$$\phi_a = \frac{Mz}{r_0^3} = -M\frac{\partial \frac{1}{r_0}}{\partial z} \qquad \qquad (186)$$

and inside the magnetic sphere it is:

$$\phi_i = \frac{Mz}{R^3} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (187)$$

where we denote the magnetic moment of the whole sphere by:

$$M = \frac{4R^3\pi}{3} \cdot \boldsymbol{M} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (188)$$

By (169) this also gives us the field-intensity $\boldsymbol{H}$. For external points the uniformly magnetized sphere acts exactly like a magnetic dipole having the magnetic moment of the sphere; at internal points, on the other hand, the field-strength is constant, that is, the field is homogeneous and:

$$\boldsymbol{H}_i = -\frac{\partial \phi_i}{\partial z} = -\frac{M}{R^3} = -\frac{4\pi}{3} \cdot \boldsymbol{M} \quad . \quad . \qquad (189)$$

that is, $\boldsymbol{H}_i$ is proportional to the magnetization and—an important and remarkable feature—is independent of the radius of the sphere. The direction of the lines of force is from positive to negative values of z, both inside and outside the sphere.

On the other hand, we have for the induction, by (173):

$$\boldsymbol{B}_a = \boldsymbol{H}_a, \; \boldsymbol{B}_i = \frac{8\pi}{3} \cdot \boldsymbol{M} \quad . \quad . \quad . \quad . \qquad (190)$$

Hence the direction of the lines of induction inside the magnet is opposed to that of the lines of force, running from negative to positive values of z.

We can deal in the same way with the case where the sphere has a concentric spherical hollow space in its interior; among other results we find here that the magnetic action of a uniformly magnetized hollow sphere is zero in the vacant space in its interior.

§ 38. The fact that a uniformly magnetized sphere generates a uniform field in its interior allows us to generalize the above theorems in a simple way for the case that the sphere, besides being permanently mag-

netized as above—we shall denote this magnetization from now on by M_p—also has a *temporary magnetization* of arbitrary, but constant susceptibility κ and a corresponding permeability μ in accordance with equation (178) or (178*a*). In the interests of greater generality we shall now suppose the sphere not to be situated in a magnetically neutral vacuum, but in a homogeneous magnetic field parallel to the z-direction, so that at an infinite distance from the sphere there is a given constant field-strength H_0 parallel to the z-axis. Let the permeability of the surrounding medium be μ_0. We can then satisfy all the conditions for the magnetostatic field if, generalizing the equations (186) and (187), we write for the magnetic potential at points outside the sphere:

$$\phi_a = \frac{C_0 z}{r^3_0} - H_0 z \quad . \quad . \quad . \quad . \quad (191)$$

and at points inside the sphere:

$$\phi_i = Cz. \quad . \quad . \quad . \quad . \quad . \quad . \quad (192)$$

The values of the two constants C_0 and C result from the boundary conditions at the surface of the sphere, namely, in the first place, the condition that the potential ϕ itself is continuous:

$$\frac{C_0}{R^3} - H_0 = C \quad . \quad . \quad . \quad . \quad . \quad (193)$$

and, in the second place, the condition that the normal component B_ν of the induction is continuous, that is, by (178*a*):

$$(\mu_0 H_a)_{r_0} = (4\pi M_p + \mu H_i)_{r_0}$$

or:

$$-\mu_0\left(\frac{\partial \phi_a}{\partial r_0}\right)_R = 4\pi M_p \cdot \frac{z}{R} - \mu\left(\frac{\partial \phi_i}{\partial r_0}\right)_R.$$

Consequently:

$$\frac{2\mu_0 C_0}{R^3} + \mu_0 H_0 = 4\pi M_p - \mu C. \quad . \quad . \quad (194)$$

If we calculate the values of the constants C_0 and C from (193) and (194), we get for the magnetic potential

at points outside and inside the sphere, by (191) and (192):

$$\phi_a = \frac{4\pi M_p + (\mu - \mu_0) H_0}{\mu + 2\mu_0}\left(\frac{R}{r_0}\right)^3 z - H_0 z \quad . \quad (195)$$

$$\phi_i = \frac{4\pi M_p - 3\mu_0 H_0}{\mu + 2\mu_0} \cdot z \quad . \quad . \quad . \quad . \quad . \quad . \quad (196)$$

and so we have obtained all the properties of the magnetic field. For external points the sphere acts like a magnetic dipole; for internal points the field is homogeneous, the field-intensity being:

$$H_i = -\frac{4\pi M_p - 3\mu_0 H_0}{\mu + 2\mu_0} \quad . \quad . \quad . \quad . \quad (197)$$

the induction (178*a*):

$$B_i = \frac{8\pi M_p + 3\mu H_0}{\mu + 2\mu_0} \cdot \mu_0 \quad . \quad . \quad . \quad (198)$$

and the magnetization (178):

$$M = \frac{1 + 2\mu_0}{\mu + 2\mu_0} M_p + \frac{3\mu_0(\mu - 1)}{4\pi(\mu + 2\mu_0)} H_0 \quad . \quad . \quad (199)$$

Let us consider some special cases. For a rigid magnet we have $\mu = 1$ and M always equal to M_p.

If the external field H_0 vanishes, the magnet acts for external points like a magnetic dipole of moment:

$$\frac{4\pi R^3 M_p}{\mu + 2\mu_0} = \frac{3M_p}{\mu + 2\mu_0} \quad . \quad . \quad . \quad . \quad (200)$$

where M_p, as in (188), denotes the permanent moment of the whole sphere. At external points the potential is then:

$$\phi_a = \frac{3M_p}{\mu + 2\mu_0} \cdot \frac{z}{r_0^3} \quad . \quad . \quad . \quad . \quad . \quad (200a)$$

For $\mu = \mu_0 = 1$ we have the case of a rigid magnet in a vacuum, which was treated in the previous paragraph. Whenever the susceptibility is positive, either in the magnet or in the surrounding medium, it weakens the field excited by the permanent magnetism, but the susceptibility of the surroundings has a stronger influence than that of the magnet.

If, on the other hand, there is no permanent magnetism present, that is, if the sphere is magnetically soft, the lines of force and the lines of induction everywhere coincide. The action of the sphere, determined by ϕ_a, at external points is in accordance with the sign of $\mu - \mu_0$. That is, according as the permeability of the sphere is greater or less than that of the surroundings the sphere tends to weaken or to strengthen the homogeneous magnetic field $\boldsymbol{H}_0$ which excites it. In the limiting case $\mu = \mu_0$ the field is not altered by the sphere at all, as is readily understood.

In the interior of the sphere the intensity of field is:

$$\boldsymbol{H}_i = \frac{3\mu_0}{\mu + 2\mu_0}\boldsymbol{H}_0 \quad . \quad . \quad . \quad . \quad . \quad (201)$$

the induction is:

$$\boldsymbol{B}_i = \frac{3\mu\mu_0}{\mu + 2\mu_0}\boldsymbol{H}_0 \quad . \quad . \quad . \quad . \quad . \quad (202)$$

and the magnetisation is:

$$\boldsymbol{M} = \frac{3\mu_0(\mu - 1)}{2\pi(\mu + 2\mu_0)} \cdot \boldsymbol{H}_0 \, . \quad . \quad . \quad . \quad . \quad (203)$$

For the most important case $\mu_0 = 1$ the magnetization of the sphere is:

$$\boldsymbol{M} = \frac{3(\mu - 1)}{4\pi(\mu + 2)}\boldsymbol{H}_0 = \frac{\kappa}{1 + \frac{4}{3}\pi\kappa}\boldsymbol{H}_0 \quad . \quad . \quad (204)$$

The ratio of the magnetization $\boldsymbol{M}$ to the exciting external intensity of field $\boldsymbol{H}_0$ must not be confused with the ratio of $\boldsymbol{M}$ to the internal intensity of field $\boldsymbol{H}_i$, which, by (175), represents the susceptibility κ. The difference between $\boldsymbol{H}_0$ and $\boldsymbol{H}_i$ is due to the fact that the sphere, no matter how small it is, produces in virtue of its magnetization a field-strength of finite magnitude in its interior. For when $\mu_0 = 1$ we have:

$$\boldsymbol{H}_i = \boldsymbol{H}_0 - \frac{\mu - 1}{\mu + 2}\boldsymbol{H}_0 = \boldsymbol{H}_0 - \frac{4\pi}{3}\boldsymbol{M}. \quad . \quad (205)$$

Thus the magnetization of the sphere causes a weakening of the field-strength in its own interior by the amount $\frac{4\pi}{3}M$. Hence this phenomenon is also called "self-demagnetization" and the numerical factor $\frac{4\pi}{3}$ is called the "self-demagnetizing factor" of the sphere. Although its value is independent of the size of the sphere, it changes with the form of the magnetic body. In general, the field-strength, and consequently also the magnetization, are not constant in the interior of a body magnetized by a homogeneous field, even if the body is very small. Only ellipsoids which are magnetized in the direction of an axis exhibit this simple behaviour. For this reason an ellipsoidal form is given to test bodies which are prepared from a substance for the purpose of determining its magnetic properties.

For an infinitely great value of the permeability or the susceptibility we get from (201) that $H_i = 0$, $\phi_i =$ const. and from (205): $M = \frac{3}{4\pi}H_0$.

Such a body, which is "absolutely soft" in magnetic respects, exhibits the maximum self-demagnetization, and this wholly compensates the exciting field-strength H_0. It behaves in every magnetic field like a conductor in an electric field, fully in accord with the laws which hold for an insulator of infinitely great dielectric constant ϵ (§ 17). Hence soft iron exerts a more or less complete "screening action," depending on the value of μ. This screening action is used, for example, in the so-called "*Panzergalvanometer*" to eliminate the disturbing effect of the earth's magnetic field.

CHAPTER IV

PONDEROMOTIVE ACTIONS IN THE STATICAL FIELD

§ 39. In the preceding pages we have devel[illegible] theory up to a certain point, but we have nowher[illegible] a method of actually measuring any of the quant[illegible] far considered which would enable us to subject the[illegible] to an experimental test. Since every meas[illegible] reduces in the last instance to a mechanical act[illegible] must, if the theory is to fulfil its true purpose, [illegible] a way of representing the mechanical or ponder[illegible] effects in a statical field. To solve this problem [illegible] pletely we need only go back to the originally int[illegible] definitions of the electric and magnetic quantit[illegible] which have their origin in the concept of energy [illegible] from mechanics.

Let us begin with the *electrostatic field*. By (2) and [illegible] the electric energy in any arbitrary system of [illegible] conductors and insulators is :

$$U = \frac{1}{8\pi}\int \epsilon E^2 d\tau = \frac{1}{8\pi}\int \epsilon\,(\text{grad}\,\phi)^2 d\tau \quad . \quad . \qquad [illegible]$$

or, integrating by parts, we have by (59) and (62) :

$$U = \frac{1}{2}\int h\phi d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \qquad [illegible]$$

which is to be integrated over all the surfaces of separ[illegible] of the conductors and insulators. Introducing the density of charge h' from (88) we can also, by (73), wr[illegible]

$$U = \frac{1}{2}\iint \frac{hh'}{r} d\sigma d\sigma' \quad . \quad . \quad . \quad . \quad . \qquad [illegible]$$

where r denotes the distance between any two surface-elements $d\sigma$ and $d\sigma'$. It is to be noted that every surface-element of the system appears twice—namely, once as $d\sigma$ and once as $d\sigma'$.

If the conductors are all embedded in a common insulator of dielectric constant ϵ, then by (92) $h' = \frac{h_1}{\epsilon}$, where we add the suffix 1 to the symbol h to prevent confusion with the density of charge h which refers to the surface-element $d\sigma$. Hence we have:

$$U = \frac{1}{2\epsilon} \cdot \iint \frac{hh_1}{r} d\sigma d\sigma_1 \quad . \quad . \quad . \quad . \quad (209)$$

where every combination of surface-elements two at a time occurs twice in the double integral. The terms of the double integral which correspond to infinitely small values of r—so that $d\sigma$ and $d\sigma_1$ coincide for them—contribute only a vanishingly small amount to the value of U.

If we write e and e_1 respectively for the elementary charges $hd\sigma$ and $h_1 d\sigma_1$, and also agree that every combination of two charges is to occur only once, the factor $\frac{1}{2}$ in (209) drops out, and we get:

$$U = \frac{1}{\epsilon} \sum\sum \frac{ee_1}{r} \quad . \quad . \quad . \quad . \quad . \quad (210)$$

This double sum is the total potential of all the elementary charges with respect to one another or the "self-potential" of the whole system of charges (I, § 104), since the self-potential of each separate elementary charge is vanishingly small.

The different expressions for the electric energy contained in the last five equations are all completely equivalent to each other in an electrostatic field, whereas in dynamic fields only the first expression (206) is valid.

§ 40. Given the values of the electrical energy, the principle of the conservation of energy leads us to the law which governs the mechanical actions of electric fields.

For let us consider a system of charged conductors and insulators moving in any way whatsoever, but so slowly that at every moment and for every position of the bodies the field may be regarded as electrostatic. Such a field is called "quasi-statical." In practice this limitation does not signify much. For the time which the electric charges take to distribute themselves on the surfaces of the conductors corresponding to the instantaneous position of the bodies is vanishingly short compared with the times in which an appreciable change of position occurs in the bodies.

If we disregard all other forces, including that of gravitation, there are only two kinds of energy: the kinetic energy K and the electrical energy U; and according to the principle of energy I (388) we have for an infinitely short interval of time:

$$d(K+U)=0.$$

On the other hand, by I (386), the change of *vis viva* is equal to the sum of the mechanical or ponderomotive work of all the forces:

$$dK=A.$$

Hence it follows that:

$$A=-dU \quad . \quad . \quad . \quad . \quad . \quad . \quad (211)$$

That is, *whenever the positions of the charged conductors and insulators of a system are altered in any arbitrary way the total work of all the ponderomotive forces produced by the charges is equal and opposite to the change of energy caused by the change in the positions of the bodies.* Hence the forces always act in a way tending to diminish the energy. Since the changes of position can be chosen at will, this theorem contains the general law of ponderomotive actions in statical fields. To derive the force from the work in any particular case we have only to take into consideration the corresponding displacement.

§ 41. As a first application of this law we determine the mechanical force which a material point charged with

the quantity of electricity e, say the little test sphere which we used in § 2, experiences in an insulating medium of dielectric constant ϵ when an electrostatic field is produced in this medium by a number of other charged points. For this purpose we must displace the reference-point carrying the charge e to an infinitesimal extent, while all the charges which produce the field and which we shall denote by e_1 remain at rest. We calculate the change in U caused by this displacement. Then, in the double sum (210) only those terms come into account which contain the factor e, since the distances between any two of the charges that generate the field remain constant. So we may write:

$$U = \frac{1}{\epsilon} \cdot e \cdot \sum \frac{e_1}{r} + \text{const.} \quad . \quad . \quad . \quad (212)$$

In this application of the formula (210) we must, however, remember that for a finite charge e in an infinitely small body the self-potential is not vanishingly small, but, on the contrary, infinitely great (positive). In view of (73) we may write in place of (212):

$$U = e \cdot \phi + \text{const.} \quad . \quad . \quad . \quad . \quad (213)$$

and we then obtain from (211) that the mechanical work performed during a displacement of the reference-point x, y, z carrying the charge e is:

$$A = -e \cdot d\phi = -e\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right).$$

If, on the other hand, we denote the mechanical force which the charge e experiences in the field by F, then:

$$A = F_x dx + F_y dy + F_z dz.$$

A comparison of the last two equations gives us:

$$F = -e \operatorname{grad} \phi = eE. \quad . \quad . \quad . \quad (214)$$

That is, *the force is equal to the product of the charge and the field-strength* and is independent of the dielectric contants.

This law gives the mechanical force exerted by an electric field on a point-charge at rest in it, and applies quite generally also to arbitrary dynamical fields. For the electric field is completely characterized by the field-strength $\boldsymbol{E}$; and it makes no difference whence the field comes and whether it is statical or not.

Since in § 2 we identified the direction of the field-strength $\boldsymbol{E}$ with the direction of the force $\boldsymbol{F}$ acting on the test-sphere for the case where the test-sphere acquired the electrification of the cat's skin, this electricity is positive, whereas that of the ebonite rod is to be written as negative in calculations.

With the help of (214) we are now also able to express the absolute value of the charge e and the field-strength $\boldsymbol{E}$ in absolute mechanical measure. For if the field is produced by a single pole charged with the amount of electricity e_1 and distant r from the test-sphere, the potential at the distance r from the pole is:

$$\phi = \frac{e_1}{\epsilon r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (215)$$

The field-strength is:

$$|\boldsymbol{E}| = \frac{e_1}{\epsilon r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (216)$$

The energy of the field is:

$$U = \frac{ee_1}{\epsilon r} + \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (217)$$

and the mechanical force on the test-sphere e is:

$$|\boldsymbol{F}| = \frac{ee_1}{\epsilon r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (218)$$

Since the force always acts in a sense tending to diminish U, it is a repulsive force for similar charges and an attractive force for dissimilar charges (Coulomb's Law).

If the two spheres are placed in another insulating medium of greater dielectric constant the force becomes smaller, since e and e_1 remain unaltered. The value of $\boldsymbol{F}$ is greatest for a vacuum.

According to Coulomb's Law (218) the unit quantity of electricity is defined in Gauss's system by the fact that it exerts a force of 1 dyne (I, § 9) on an equal quantity of electricity distant 1 centimetre from it. In other systems of measurement the definition is less simple. It is useful to form a picture of the units corresponding to a definite system. In Gauss's system the unit quantity of electricity is so small in practice that a few hundred units can be generated by rubbing an ebonite rod a few times. With regard to potential, an ordinary Wimshurst machine can bring a conductor to a potential of about 200 [$\text{grm.}^{\frac{1}{2}}\ \text{cm.}^{\frac{1}{2}}\ \text{sec.}^{-1}$]. On the other hand, the galvanic potential of a Daniell cell (163) is only about 0·0035 in the same units, and the Voltaic potential (148*a*) between zinc and copper is of the same order, but a little less in value.

§ 42. We have become acquainted with the law of the ponderomotive actions of electrostatic fields in two totally different forms, each of which has its characteristic advantages and disadvantages, so that in some cases one form is used, and in other cases the second form is preferable. The *Potential* or *Integral Law* is the first form (211). It gives the total work of all the ponderomotive forces when any changes of position are effected; it is therefore particularly useful if we wish to combine all the active forces in a single resultant. For the potential or the energy U of the whole field can often be represented by a comparatively simple expression which allows us, for example, to see immediately whether it increases or decreases during the change of position in question. The sense of the direction of the resulting mechanical action can also then be seen at a glance.

If, on the other hand, we wish to gain an insight into the details of the event in question, it is more expedient to apply the Differential Law of Force (214), which tells us what mechanical action there is on each individual elementary charge, but does not give us the sum-total of the actions.

This contrast between the Integral and the Differential Law will often be encountered in the sequel even in electromotive effects, and the above remarks will always apply.

Concerning the Law of Force (214) we have yet to mention a point of fundamental importance which will force itself on our attention when we apply the law. It refers to the meaning of $\boldsymbol{E}$ in this equation. For it is clear that $\boldsymbol{E}$ is not the actual strength of the electric field at the point where the charge e is situated, but rather the field-strength that would exist at that point if the charge e were not present at all. For the charge e contributes nothing to the value of the potential function ϕ in (214). The method of probing a field by means of a testing-sphere has a meaning only if we may assume that the field is not changed, or at least not appreciably, by the introduction of the testing-sphere. But if we imagine a finite charge e in an infinitely small body, this charge, indeed, produces an infinite field-strength in the body, and this must not be added to $\boldsymbol{E}$. This is easy to understand if we consider that it is impossible for the charge e to set itself into motion.

No difficulty of this kind occurs in the case of the Integral Law (211). Hence in doubtful cases it is advisable to fall back on this law before making our decisions. In the case of a point-charge e, however, the matter is quite simple. We have only to imagine the charge e absent, and then to take for the field-strength $\boldsymbol{E}$ that which is left at the point in question.

The problem becomes more complicated if we wish to find the mechanical force which acts on the surface-element of a conductor which carries the charge $hd\sigma$. Although the field-strength $\boldsymbol{E}$ is no longer infinite in this case, it is discontinuous. For on the side of the insulator it is equal to $\frac{4\pi h}{\epsilon}$ by (63), whereas on the side of the conductor it is zero, and it would be equally wrong to write down either value for the expression of $\boldsymbol{E}$ in the equation (214) for $\boldsymbol{F}$.

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movable part of the plate F, then the attractive force exerted on it by the opposite plate is, by (219):

$$\frac{2\pi h^2}{\epsilon} F \quad . \quad . \quad . \quad . \quad . \quad . \qquad (220)$$

which also results directly if we calculate the attraction (87) of the fixed plate, whose size is assumed infinitely great in comparison with the distance between the plates. By measuring this force we obtain the density of charge h, and, by (60) the potential difference between the plates in absolute measure for a vacuum ($\epsilon = 1$), whereas for any other insulating medium the dielectric constant ϵ is obtained from the fact that the attractive force for a given definite charge h is inversely proportional to the dielectric constant, whereas for a definite potential difference it is directly proportional to the dielectric constant. The first case is realized when the charged plates are kept insulated, the second if they are kept connected with the poles of a definite galvanic chain.

If more generally we have any system of charged conductors, 1, 2, 3 . . . in a common insulator, as in § 21, then the field-energy U is, by (207) (since ϕ is constant in every individual conductor):

$$U = \frac{1}{2}(e_1\phi_1 + e_2\phi_2 + \ldots) \quad . \quad . \quad . \qquad (221)$$

and by either (100) or (102):

$$U = \epsilon\left(\frac{c_1}{2}\phi_1^2 + c_{12}\phi_1\phi_2 + \frac{c_2}{2}\phi_2^2 + c_{13}\phi_1\phi_3 + \ldots\right) . \qquad (222)$$

and also by (103):

$$U = \frac{1}{\epsilon}\left(\frac{c'_1}{2}e_1^2 + c'_{12}e_1e_2 + \frac{c'_2}{2}e_2^2 + c'_{13}e_1e_3 + \ldots\right) . \qquad (223)$$

For the case of a single conductor we obtain:

$$U = \frac{1}{2}e_1\phi_1 = \frac{\epsilon c_1}{2}\phi_1^2 = \frac{c'_1}{2\epsilon}e_1^2 \quad . \quad . \quad . \qquad (224)$$

If, for example, the conductor is a sphere of radius R, then, in view of the capacity (95), we have :

$$U = \frac{\epsilon R}{2} \phi_1^2 = \frac{e_1^2}{2\epsilon R} \quad . \quad . \quad . \quad . \quad (225)$$

A decrease in the radius of the insulated sphere (soap-bubble) by the amount dR causes a decrease in energy of :

$$- dU = \frac{e_1^2}{2\epsilon R^2} dR = \frac{8\pi^2 R^2 h^2}{\epsilon} \cdot dR$$

which by (211) represents the mechanical work of extension due to the electric charge. Since the increase in volume amounts to $dV = 4R^2\pi \cdot dR$, we get from II (278) for the mechanical pressure, that is, for the force on unit surface :

$$- \frac{dU}{dV} = p = \frac{2\pi h^2}{\epsilon}$$

which agrees with the general formula (219).

§ 44. The general law of force for the ponderomotive actions produced in an electrostatic field may be given in another form which has shown itself to be particularly valuable for developing Maxwell's theory. It results from the following reflection. In general mechanics we regard as the most important foundation of statics the following theorem, which depends on the law of the equality of action and reaction (Newton's third law) : if a system of points is in equilibrium, then the external forces acting on the system, which we suppose rigid, are in equilibrium (I, § 112). If we now imagine a charged conductor in any arbitrary electrostatic field to be surrounded by an insulator of dielectric constant ϵ and kept in equilibrium by appropriate mechanical forces, for example, by means of thin silk threads which are attached and kept taut, no other forces acting on the body except the mechanical action of the electric field, then, according to the above theorem, the equilibrium persists even if we remove the conductor as a point-system together with an arbitrary part of the insulator completely surrounding it and imagine the whole system to become rigid, and take into

consideration only the external forces that act on it. If we denote the surface of this point-system, which according to our assumption lies completely within the insulator, by σ, then, by the principle of contiguous action, the external forces due to the electric field are restricted to certain pressures that act on the surface-elements $d\sigma$ from without; by II, § 20 these pressures are represented in the most general case by a symmetrical tensor having the six components X_x, Y_y, Z_z, X_y, Y_z, Z_x; this is the so-called *tensor of electric pressure or potential*. The resultant of all these pressures that act on the surface-elements $d\sigma$ from without is therefore equal and opposite to the force exerted by the silk threads, or, what comes to the same thing, it is equivalent to the ponderomotive force of the electrostatic field on the conductor. For example, if we consider the x-component of the resultant and denote the surface of the conductor by s in contrast with the arbitrary surface σ in the insulator, the normal to s being n, then we obtain by II (74) and (210):

$$\int (X_x \cos(\nu x) + X_y \cos(\nu y) + X_z \cos(\nu z))\, d\sigma$$
$$= \frac{2\pi}{\epsilon}\int h^2 \cos(nx) ds \quad . \quad . \quad . \quad . \quad (226)$$

A corresponding equation holds for the y- and the z-component. The normals ν and n to the surfaces are to be taken as inwardly directed to the insulator surrounding the surfaces σ and s.

We must not, of course, regard the tensor of electric pressure as representing an ordinary mechanical pressure; one reason is that it also acts in a vacuum. Hence it is probably not possible to interpret it entirely in terms of a graphical model. But its physical meaning remains unaffected by this difficulty. Its physical sense is that its resultant action on any arbitrary closed surface in every case gives the ponderomotive force exerted by the electrostatic field on the matter in it. The position is not different in the case of energy. For however difficult

it is to form a picture of the flux of energy in a pure vacuum, great practical importance attaches to the theorem that the total energy that flows through a closed surface gives the change in the quantity of energy contained inside.

From the point of view of the principle of contiguous action we must postulate, however, that the electric pressure tensor and all its components are completely determined at every point of the field by the electrical intensity of field E at that point. Since, further, the equations (226) must hold for any arbitrary form of the surface σ, we get from them the values of the six quantities X_x, ... To derive them we first transform the surface integral in $d\sigma$ into a volume integral over the insulator enclosed by the surfaces σ and s, and a surface integral in ds:

$$-\int\left(\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\right)d\tau$$

$$-\int(X_x\cos(nx)+X_y\cos(ny)+X_z\cos(nz))\,ds \quad . \qquad (227)$$

This expression is equal to the right-hand side of (226) only if in the whole interior of the insulator we have:

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}=0 \quad . \quad . \quad . \qquad (228)$$

and if, taking (63) into account, we have everywhere at the surface of the conductor:

$$X_x\cos(nx)+X_y\cos(ny)+X_z\cos(nz)$$
$$=-\frac{\epsilon E^2}{8\pi}\cdot\cos(nx) \quad . \quad . \quad . \quad . \qquad (229)$$

If we recollect that at the surface of the conductor we always have:

$$E_x : E_y : E_z = \cos(nx) : \cos(ny) : \cos nz$$

then we satisfy the equation (229) by setting:

$$X_x=-\frac{\epsilon E^2}{8\pi}+C(E_x{}^2-E^2)$$
$$X_y=CE_xE_y,\quad X_z=CE_xE_z$$

where C stands for an arbitrary constant. If we substitute these expressions in the differential equation (228) we find that in view of (50) and (55) they are always satisfied by the value:

$$C = -\frac{\epsilon}{4\pi}$$

Hence the components of the electric pressure tensor come out as:

$$\left.\begin{aligned} X_x &= -\frac{\epsilon}{8\pi}(2E_x^2 - E^2), \quad \text{and similarly:} \\ Y_y &= -\frac{\epsilon}{8\pi}(2E_y^2 - E^2) \\ Z_z &= -\frac{\epsilon}{8\pi}(2E_z^2 - E^2) \\ Y_z &= -\frac{\epsilon}{4\pi}E_yE_z, \quad Z_x = -\frac{\epsilon}{4\pi}E_zE_x, \quad X_y = -\frac{\epsilon}{4\pi}E_xE_y \end{aligned}\right\} \quad (230)$$

With reference to the magnitudes of the three principal pressures and directions of the axes of the principal pressures (II, § 20) it is clear that one of the axes of the principal pressures coincides with the direction of the field-strength E. If we take this direction for the z-axis we get:

$$E_x = 0, \quad E_y = 0, \quad E_z^2 = E^2.$$

Consequently

$$\left.\begin{aligned} X_x = Y_y = \frac{\epsilon}{8\pi}E^2, \quad Z_z = -\frac{\epsilon}{8\pi}E^2 \\ Y_z = Z_x = X_y = 0 \end{aligned}\right\} \quad . \quad (231)$$

On account of the equations in the last two lines the x-axis and the y-axis are also axes of the principal pressures. Thus in the direction of the field-strength E the electric pressure tensor gives a *tension* equal in value to the density of electric energy (1), but at every direction at right angles to this pull it gives a *pressure* of the same value. All the ponderomotive effects of an electrostatic

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field can likewise be deduced from this simple theorem. But it reaches far beyond electrostatics and holds quite generally for any electric field whatsoever, for the same reason as that given to account for the general validity of the law of force (214).

§ 45. If we now turn to the *ponderomotive* actions in *magnetic fields*, we find that here exactly the same considerations apply as in the case of the electric fields. As the fundamental law, we again obtain the law of potential in the form of (211), where U now denotes the magnetic energy of the field. From this there again follows Coulomb's Law of Force (218) for the attraction of two unlike magnetic poles or the repulsion of two like magnetic poles.

In accordance with the remarks in § 36, an isolated positive or negative magnetic pole can be realized by means of a long thin uniformly magnetized needle.

Using the small magnetic needle described in § 3 for probing a magnetic field, we further consider the ponderomotive action of a given arbitrary magnetostatic field of strength $\boldsymbol{H}$ on an infinitely small rigid magnet of moment M situated in it. It will be instructive and useful to carry out the calculation in two ways, firstly by means of the law of force, secondly by means of the law of potential.

The magnet consists of two poles carrying charges $+m$ and $-m$ at the infinitely small distance l apart, so that $ml = M$. If we call the co-ordinates of the negative pole x, y, z, then those of the positive pole are $x + \alpha l$, $y + \beta l$, $z + \gamma l$, where α, β, γ are the direction cosines of the axis of the magnet. By (214) the components of the force acting on the negative pole are $-m\boldsymbol{H}_x$, $-m\boldsymbol{H}_y$, $-m\boldsymbol{H}_z$, and those acting on the positive pole are:

$$m\left(\boldsymbol{H}_x + \frac{\partial \boldsymbol{H}_x}{\partial x}\alpha l + \frac{\partial \boldsymbol{H}_x}{\partial y}\beta l + \frac{\partial \boldsymbol{H}_x}{\partial z}\gamma l\right), \text{ and so forth.}$$

The forces acting on the two poles, regarded now as

acting on the rigid body of the magnet, are, by I (366), composed of a resultant force having components:

$$F_x = ml\left(\alpha\frac{\partial H_x}{\partial x} + \beta\frac{\partial H_x}{\partial y} + \gamma\frac{\partial H_x}{\partial z}\right)$$
$$= M_x\frac{\partial H_x}{\partial x} + M_y\frac{\partial H_x}{\partial y} + M_z\frac{\partial H_x}{\partial z} \quad . \quad . \quad . \quad (232)$$

and so forth, and a resultant couple whose components are:

$$N_x = \beta lmH_z - \gamma lmH_y = M_yH_z - M_zH_y \quad . \quad . \quad (233)$$

and so forth. Or, we may write (233) in the vector form:

$$N = [M, H] \quad . \quad . \quad . \quad . \quad . \quad . \quad (234)$$

where terms of a lower order of magnitude are neglected.

If we wish to start out from the potential law we have again to consider any infinitely small displacement of the rigid magnetic dipole; we characterize this displacement by means of the three translational components u, v, w, and the three rotational components ξ, η, ζ. Then by I (349) the work performed by the ponderomotive forces of the magnetic field is:

$$A = uF_x + vF_y + wF_z + \xi N_x + \eta N_y + \zeta N_z \quad . \quad (235)$$

Considering now the change of energy U produced by the displacement, we find it expedient to take out of the expression for the total energy of the system under consideration those terms which remain unaltered during the displacement, just as was done in § 41 in deriving (213). This is accomplished by observing that the total energy consists of three terms: firstly, of the energy of the field when the dipole is imagined absent (self-potential of the field), secondly, the energy of the dipole, when the field is imagined absent (self-potential of the dipole), and thirdly, the potential of the field with respect to the dipole. Since the first two terms are not influenced by the displacement of the dipole, the third remains as the

only one which changes. We then obtain just as in (213):

$$U = -m\phi_{x,y,z} + m\phi_{x+\alpha l,\, y+\beta l,\, z+\gamma l} + \text{const.}$$

$$U = ml\left(\alpha\frac{\partial\phi}{\partial x} + \beta\frac{\partial\phi}{\partial y} + \gamma\frac{\partial\phi}{\partial z}\right) + \text{const.}$$

$$U = -(\boldsymbol{M}_x\boldsymbol{H}_x + \boldsymbol{M}_y\boldsymbol{H}_y + \boldsymbol{M}_z\boldsymbol{H}_z) + \text{const.} \quad . \quad . \quad (236)$$

The displacement brings the pole from x, y, z to $x + u$, $y + v$, $z + w$ and changes its direction to $\alpha + d\alpha$, $\beta + d\beta$, $\gamma + d\gamma$, so that by I (499):

$$d\alpha = \gamma\eta - \beta\zeta,\, d\beta = \alpha\zeta - \gamma\xi,\, d\gamma = \beta\xi - \alpha\eta \quad . \quad . \quad (237)$$

Hence, since $\boldsymbol{M}$ is constant:

$$d\boldsymbol{M}_x = Md\alpha,\; d\boldsymbol{M}_y = Md\beta,\; d\boldsymbol{M}_z = Md\gamma$$

and:

$$d\boldsymbol{H}_x = \frac{\partial \boldsymbol{H}_x}{\partial x}u + \frac{\partial \boldsymbol{H}_x}{\partial y}v + \frac{\partial \boldsymbol{H}_x}{\partial z}w$$

and so forth. Consequently by (236):

$$\begin{aligned} -dU = {} & M(\boldsymbol{H}_x d\alpha + \boldsymbol{H}_y d\beta + \boldsymbol{H}_z d\gamma) \quad . \quad . \quad . \quad (238) \\ & + u\left(\boldsymbol{M}_x\frac{\partial \boldsymbol{H}_x}{\partial x} + \boldsymbol{M}_y\frac{\partial \boldsymbol{H}_y}{\partial x} + \boldsymbol{M}_z\frac{\partial \boldsymbol{H}_z}{\partial x}\right) \\ & + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}$$

If we substitute in this the values (237) for $d\alpha$, $d\beta$, $d\gamma$, and, by (211), equate the expressions (235) and (238) for any arbitrary displacement, then we again get for the resultant force $\boldsymbol{F}$ and the resultant couple $\boldsymbol{N}$ exactly the same expressions (232) and (233) as were obtained from the law of force. This means that the value of the field-strength $\boldsymbol{H}$ is important only for the resultant couple, but not for the resultant force. A homogeneous field, no matter how intense it may be, is never able to move a magnet bodily from its position. Hence we measure the field-strength $\boldsymbol{H}$ by the torque it produces.

If a small magnetic needle which can rotate freely about its centre of gravity is situated in a magnetic field, then when it is in stable equilibrium, for which U

assumes its minimum value (I, § 119), the direction of the magnetic axis, taken from the negative to the positive pole, coincides with that of the field-strength. But since in § 3 we defined the direction of the field-strength as the direction from the south pole of the needle to its north pole when in stable equilibrium, it follows that the north pole is to be regarded as positive and the south pole as negative, and, correspondingly, the magnetism at the north pole as positive, and that at the south pole as negative.

In the case of a uniformly magnetized sphere, such as we may approximately regard the earth to be, the direction of the magnetic lines of force both inside and outside the sphere is, by § 37, from the free positive charge to the free negative charge. Now since the direction of the lines of force of the earth's magnetic field runs roughly from the geographical south pole to the geographical north pole, it follows that the geographical north pole of the earth is a magnetic south pole, that is, its free charge is negative.

We can also follow Gauss and measure magnetic quantities in the absolute system of units by combining two different experiments which are performed on a magnet needle of moment M in the earth's field of intensity H. In the first experiment we determine the time of swing of the needle in the earth's field (vibration experiment), it being assumed that the needle can rotate freely about its centre of gravity: in the second we determine the amount by which the same needle, now kept fixed, deflects any other needle which can move freely about its centre of gravity (deflection experiment). By I, § 140, the first experiment gives the turning moment $M \cdot H$, the second gives the ratio $\frac{M}{H}$, which determines the angle of deflection. From these two values we can get M and H separately. The unit of intensity of magnetic field is often called the "gauss." The horizontal component of the intensity of the earth's magnetic field amounts in

Germany to about 0·2 gauss, its deviation from the geographical meridian to about 10° to the west (declination), the magnetic dip below the horizontal direction to about 66°.

§ 46. A magnetic field exerts ponderomotive forces even on a magnetically soft, paramagnetic or diamagnetic body, as can be deduced either from the potential law or the law of force. If we wish to calculate the total effect it is preferable to use the potential law. For example, we have already seen above that a sphere that has been introduced into a magnetic field acts on the field at the place where it is situated, weakening or strengthening it, both inside and outside the sphere, according as its permeability μ is greater or less than the permeability μ_0 of the surrounding medium. It is easy to see that this theorem will hold qualitatively for every other form of surface of the body. Hence it follows that the ponderomotive forces, which always tend to diminish the energy U of the whole system, and hence also the field-intensity, would, when acting on a body which has greater permeability than the surrounding medium, tend to bring the body into those parts of the field where the field-strength is greatest; for there it would decrease the energy of the field to the greatest extent. In a homogeneous field, however, it does not matter at all to the energy where the body is situated, and hence no ponderomotive forces act on it at all as a whole. If, conversely, $\mu < \mu_0$, the ponderomotive forces drive the bodies to the points where the field-strengths are weakest.

Hence if a rod of any substance suspended so that it can rotate in a horizontal plane is placed midway between the poles of a powerful horse-shoe magnet, it will set itself axially in the direction of the line connecting the poles or transversely, that is, perpendicular to this line, according as the permeability of the substance of which the rod is composed is greater or less than that of the gas or liquid in which the magnet and the rod are immersed. This is not to be explained by saying that in the one case

the rod always sets itself in the direction of the lines of force and in the other at right angles to them, but on the ground that in the direction of one of the poles the intensity of field increases more rapidly than in a lateral direction. If we arrange so that the field-strength increases more rapidly in the direction perpendicular to the lines of force than in the direction parallel to them, the reverse effect is observed.

The same considerations apply to electrostatics and explain why an electrically uncharged pith-ball is attracted by a rod of ebonite that has been rubbed; the reason is that the dielectric constant of the pith-ball is greater than that of the surrounding air.

If we are more interested in the total resulting ponderomotive effect than in the individual local forces, the law of force is often preferable to the potential law. The law of force is given in its most general form by the expression for the tensor of magnetic pressure or potential. This runs, analogously to 230 :

$$\left.\begin{aligned} X_x &= -\frac{\mu}{8\pi}(2H_x^2 - H^2) \\ Y_y &= -\frac{\mu}{8\pi}(2H_y^2 - H^2) \\ Z_z &= -\frac{\mu}{8\pi}(2H_z^2 - H^2) \\ Y_z &= -\frac{\mu}{4\pi}H_yH_z,\ Z_x = -\frac{\mu}{4\pi}H_zH_x,\ X_y = -\frac{\mu}{4\pi}H_xH_y \end{aligned}\right\} \quad (230)$$

When applied to any arbitrarily small closed surface this tensor gives the ponderomotive force of the magnetic field which acts on the matter inside the surface, in the manner of the equations II (83).

CHAPTER V

STATIONARY ELECTROMAGNETIC FIELD

§ 47. The next step takes us from the statical electric and magnetic fields so far considered to the stationary fields. These differ from the statical fields in that, although the electromagnetic state of the field in their case undergoes no changes with the time at any point, nevertheless a continual transformation of electric energy into heat can take place at certain points. The equations (54) and (55) then continue to remain valid, that is:

$$\operatorname{curl} E = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (240)$$

and:

$$\operatorname{curl} H = \frac{4\pi}{c} J \quad . \quad . \quad . \quad . \quad . \quad (241)$$

but the product κE^2 or $\kappa E = J$ can differ from zero; that is, in a conductor a flux can exist which is everywhere constant in magnitude and direction.

A case of this kind occurs, for example, when a number of conductors of the first and the second class form a multiply connected space or a closed chain (§ 31). The law of potential sequence (164) is then no longer obeyed. Instead, a relation of the following form holds:

$$E_{12} + E_{23} + \; . \; . \; . \; + E_{n-1,\,n} + E_{n1} = E \quad . \quad (242)$$

where E, the electromotive force of the chain, is a quantity which differs from zero and is determined by the chemical nature of the conductors. As we have seen, in this case the conditions for a statical state can no longer be fulfilled, and hence certain currents will arise, which must finally assume a stationary character if the external conditions are to be maintained. We shall now investigate stationary states of this type.

The chief distinction between stationary fields and the statical fields that were considered above is that in the former, owing to the relation (241), the electric and magnetic quantities are no longer independent of each other; rather, wherever electric currents exist magnetic fields also occur. We shall now set out to find the electric field as well as the electric currents for a given case and to investigate why also the magnetic field is completely determined by the electric currents.

§ 48. Let us imagine, as above, a closed galvanic chain consisting of any number of conductors of the first and the second class which form a doubly connected space with one another and are surrounded by a common insulator. We shall suppose the contact potentials at the boundary of each pair of substances to be given, so that the electromotive force (242) of the whole chain is also known.

By (240) the electric intensity of field E has in the stationary state a potential ϕ inside and outside the conductors, and this potential is one-valued, because every arbitrary closed curve can be continuously contracted right down to zero without leaving the domain in which (240) holds (cf. II, § 69). On the other hand, the function ϕ is discontinuous, for at the boundaries of each pair of substances it undergoes abrupt changes of given amounts.

In general the first differential coefficients of ϕ are continuous. Discontinuities occur only at the boundary surfaces; and on account of the stationary state we always have, by (45):

$$J_\nu + J'_{\nu'} = 0$$

or:

$$\kappa \frac{\partial \phi}{\partial \nu} + \kappa' \frac{\partial \phi'}{\partial \nu'} = 0 \quad . \quad . \quad . \quad . \quad . \quad (242a)$$

which, in the case of a boundary surface between a conductor and an insulator ($\kappa = 0$), reduces to the simpler relation:

$$\frac{\partial \phi}{\partial \nu} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (243)$$

This can be expressed in simple physical language by saying that the electric charges at the boundary surfaces are not affected by the currents.

Finally, by (50), the electric potential function ϕ satisfies Laplace's equation:

$$\Delta\phi = 0 \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (244)$$

everywhere in the interior of the conductors and the insulator.

On the basis of the above conditions it can now be proved directly that the electric intensity of field and also the electric current in the conductors are fully determined.

For if we assume that there are two different functions, ϕ and ϕ', which may both be regarded as electric potential functions of the system under consideration and set their difference $\phi' - \phi = \phi_0$, then the function ϕ_0 also fulfils the conditions (242), (243) and (244). Moreover, ϕ_0 is not only one-valued, but also continuous because in it the abrupt changes of ϕ and ϕ' compensate each other.

We now write down for any individual conductor the well-known transformation of the integral:

$$\int d\tau\,\kappa\left\{\left(\frac{\partial\phi_0}{\partial x}\right)^2 + \left(\frac{\partial\phi_0}{\partial y}\right)^2 + \left(\frac{\partial\phi_0}{\partial z}\right)^2\right\} = -\int d\tau\kappa\phi_0\cdot\Delta\phi_0 - \int d\sigma\kappa\phi_0\cdot\frac{\partial\phi_0}{\partial\nu} = -\int d\sigma\kappa\phi_0\cdot\frac{\partial\phi_0}{\partial\nu}$$

and then form the sum of the corresponding integrals for all the conductors. We obtain as the total sum Q of all these positive space-integrals the expression:

$$Q = -\Sigma\int d\sigma\cdot\left(\kappa\phi_0\cdot\frac{\partial\phi_0}{\partial\nu} + \kappa'\phi'_0\frac{\partial\phi'_0}{\partial\nu'}\right) \quad . \quad (245)$$

The summation Σ is to be effected over all the boundary surfaces of each pair of neighbouring conductors, whereas, on account of (243), the terms that refer to the insulator boundary are omitted.

Now since ϕ_0 is continuous and equal to ϕ'_0, $Q = 0$ by (242a) and ϕ_0 is constant everywhere in all the conductors. Hence two different potential functions ϕ and ϕ' of the

system can differ only by a constant, so that the electric intensity of field $\boldsymbol{E}$ and the electric current $\boldsymbol{J} = \kappa \boldsymbol{E}$ are everywhere completely determined. The additive constant which still remained undetermined in the value of the potential function ϕ can only be specified definitely if either the total charge on the whole system of conductors is known or if a definite potential is given to one of the conductors, say, by connecting it to earth. If we assume this additive constant to be fixed in this manner, then the potential in the surrounding insulator can be calculated in the well-known way (§ 19), so that all the electric charges can be determined and the whole problem is solved. We see that neither nature of the insulator nor the statical charges are of any consequence for the properties of the electric field and the current in the conductors, but only the constant potentials and the conductivities. By (242a) the conductivities κ play exactly the same part in deciding the course of the lines of force in the conductors as do the dielectric constants ϵ for the lines of force at the boundary of two insulators or the permeabilities μ at the boundary of two magnetizable substances. The field-strength there always undergoes a characteristic abrupt change both in magnitude and in direction (§ 17).

§ 49. We shall now consider more closely the properties of the electric currents that have been shown to exist in the conductors. Since the lines of flow coincide with the electric lines of force, they intersect the equipotential surfaces ϕ = const. in the direction of the normals, as in the statical field, whereas they run tangentially along the surface of the insulator. All stream-lines that start out from the points of an arbitrarily situated surface-element $d\sigma$ form, when taken together, a "current filament" within which the electricity flows like an incompressible liquid in a closed tube, since, on account of the stationary state, the same amount of electricity:

$$d\sigma \, . \, J_\nu = - \, d\sigma \, . \, \kappa \, . \, \frac{\partial \phi}{\partial \nu} \quad . \quad . \quad . \quad (246)$$

flows through every cross-section of the filament in unit time. Here ν denotes the normal of $d\sigma$ in the direction of the current. At the boundary of two conductors the current-filament in general changes its direction suddenly without the quantity (246), the intensity of the current-filament, changing.

The total flow of electricity in the system of conductors is composed of infinitely thin current-filaments of this kind, and the integration of all the intensities (246) over the whole cross-section of a conductor gives the "intensity of current" or the "current-strength" in the system of conductors:

$$J = -\int \kappa \frac{\partial \phi}{\partial \nu} d\sigma \quad . \quad . \quad . \quad . \quad . \quad (247)$$

which represents the quantity of electricity which flows in unit time through any cross-section of the system of conductors, independently of the form and position of the cross-section.

Let us next consider the case of *linear* conductors, by assuming all the cross-sections of the conducting systems to be infinitely thin. The whole current then reduces to a single infinitely thin current-filament, and we can take $d\sigma$ to stand for the normal cross-section q, thus $d\nu$ denotes the length ds of a conductor. Omitting the integral sign in (247) we may write:

$$\frac{\partial \phi}{\partial s} = -\frac{J}{\kappa q}$$

and, integrating with respect to s from one cross-section 1 to another cross-section 2 of the same conductor, we get:

$$\phi_1 - \phi_2 = \frac{J}{\kappa}\int_1^2 \frac{ds}{q} = J \, . \, w \quad . \quad . \quad . \quad . \quad (248)$$

The constant:

$$w = \frac{1}{\kappa}\int_1^2 \frac{ds}{q} \quad . \quad . \quad . \quad . \quad . \quad . \quad (249)$$

which occurs here is called the "resistance" of the piece of the conductor between 1 and 2. If q is constant, the

resistance is proportional to the length and inversely proportional to the cross-section of the piece of conductor. The coefficient $\frac{1}{\kappa}$ is called the "specific resistance" of the substance of which the conductor is composed. By (248) the intensity of current J is equal to the quotient of the fall of potential by the resistance.

If we sum up the equation (248) over various conductors in succession, by writing down the corresponding relation at first for the first conductor of cross-section 1 to the boundary of the second substance, then from the second substance to the boundary of the third substance and so forth up to an arbitrary cross-section n of the nth conductor, and then adding up all the equations so obtained, we get, taking into account the discontinuity (§ 27) of the potential function ϕ at the boundary of two conductors:

$$\phi_1 + E_{12} + E_{23} + \ldots E_{n-1,n} - \phi_n$$
$$= J \, . \, (w_1 + w_2 + \ldots + w_n) \quad . \quad . \quad (249a)$$

or:

$$J = \frac{\phi_1 - \phi_n + \Sigma E}{\Sigma w} \quad . \quad . \quad . \quad . \quad . \quad (250)$$

where ΣE denotes the sum of the contact potentials or the electromotive forces that lie between the cross-sections 1 and n, and Σw denotes the sum of the resistances that lie between 1 and n.

If, finally, we pass right round the closed chain, the cross-section n coincides with the cross-section 1, and since the electric potential function ϕ is one-valued, we have $\phi_n = \phi_1$, and obtain from (250):

$$J = \frac{\Sigma E}{\Sigma w} \quad . \quad . \quad . \quad . \quad . \quad . \quad (251)$$

This is Ohm's Law. It states that the intensity of current in a closed system of conductors is equal to the quotient of the total electromotive force by the total resistance of the current circuit.

§ 50. In *spatially* extended conductors, which we now pass on to consider, the conditions are much more complicated, in that in this case the positions of the current-filaments in a conductor are not given at the outset, but have in general first to be calculated by integrating the field-equations. But the expressions (246) and (247) always hold for the current-intensities in an individual current-filament and in the total system of conductors. Hence the equation (250) also holds if it is applied to an individual infinitely thin current-filament.

If we now add up the current-intensities of all the current-filaments, which lying together laterally traverse the system of conductors, from the equipotential surface $\phi = \phi_1$ in the conductor 1 to the equipotential surface $\phi = \phi_n$ in the conductor n, then we obtain from (250) for the total intensity of current of the system :

$$J = (\phi_1 - \phi_n + \Sigma E) \cdot S \frac{1}{\Sigma w} \quad . \quad . \quad . \quad (252)$$

In contrast with the summation Σ over the parts of a current-filament that lie "behind" one another or "in series," the summation here is to be performed over the current-filaments that lie "adjacent" to one another or "in parallel," so we distinguish the latter summation by using the letter S.

If we follow the analogy of (250) and write :

$$J = \frac{\phi_1 - \phi_n + \Sigma E}{W} \quad . \quad . \quad . \quad . \quad (253)$$

we can call the quantity :

$$W = \frac{1}{S \frac{1}{\Sigma w}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (254)$$

the resistance of the spatial system of conductors in question. The denominator of this expression is also called the "conductivity" (*Leitwert*) of the system. For linear conductors W becomes equal to Σw. The manner in which the summations are to be effected can be character-

ized by saying that in summing over spaces that are traversed in series the resistances must be added, whereas in summing over spaces that are traversed in parallel the conductivities must be added. But this rule is of practical use for calculating W only when the position of the current-filaments is known. We can speak of the resistance of a spatially extended conductor in a definite sense only if we know in what way it is traversed by the current.

In applying this result to the total closed circuit, ϕ_n again coincides with ϕ_1, and by (253) we obtain Ohm's Law also for space currents:

$$J = \frac{\Sigma E}{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad (255)$$

§ 51. As an example let us calculate the resistance W of an electrolytically conducting liquid or electrolyte (salt-solution) in the form of a hollow circular cylinder with the radii $a < b$ and height h, which is bounded by two coaxial metal cylinders, the electrodes, and is traversed by the current from the inside outwards. The inner electrode is then called the anode, the outer electrode the cathode; and the potential ϕ in the electrolyte sinks from the value ϕ_1 at the inner surface to a value ϕ_2 at the outer surface. The connection of the cathode with the anode to complete the circuit can be supposed to be effected around the outside of the system, but in such a way that the symmetrical character of the current-flow is not prejudiced.

Since metals have a much greater conductivity κ than electrolytes we can as a rule, on account of the boundary condition (242a), neglect the potential gradient in the metals compared with that in the electrolytes. The potential inside a metal is then essentially constant and the surface of each electrode is an equipotential surface. The potential ϕ, and hence the whole current-flow in the electrolyte, is then fully determined by the constant value of ϕ at the electrodes.

In the present case the expression for the potential

function ϕ in the electrolyte can easily be calculated. On account of the boundary conditions (243) at the highest and lowest cross-section of the cylinders, where the electrolyte is in contact with the surrounding insulator and on account of the symmetry about the axis of the cylinder, which we shall take as our z-axis, *all the current-lines follow a radial course.* Hence the potential depends only on the distance ρ of the reference-point from the axis, and the general integral of Laplace's equation (244) is represented by the logarithmic potential (I (146)) plus a constant:

$$\phi = A \log \rho + B \quad . \quad . \quad . \quad . \quad (256)$$

The values of the two integration constants A and B follow from the condition that for $\rho = a$, $\phi = \phi_1$ and for $\rho = b$, $\phi = \phi_2$. Hence the potential fall from the boundary of the anode to the boundary of the cathode is:

$$\phi_1 - \phi_2 = - A \log \frac{b}{a}. \quad . \quad . \quad . \quad (257)$$

On the other hand, we obtain the total intensity of current J from (247) by integrating over an arbitrary coaxial cylinder which serves as a cross-section of the conductor, being of radius ρ and height h: this gives:

$$J = - \int \kappa \frac{\partial \phi}{\partial \rho} d\sigma.$$

If we substitute the expression (256) for ϕ and the value $2\pi\rho \,.\, dz$ for $d\sigma$ and integrate from 0 to h with respect to z we get:

$$J = - 2\pi\kappa A h$$

Dividing by (257) and using (253) we obtain the required value:

$$W = \frac{\phi_1 - \phi_2}{J} = \frac{1}{2\pi\kappa h} \cdot \log \frac{b}{a}. \quad . \quad . \quad (258)$$

Thus the resistance is inversely proportional to the height of the cylinder and depends, remarkably enough, only on the ratio of the two limiting radii.

We can arrive at the same value of W for the above case in a more direct way by means of equations (254) and (249) if we make use of the fact that the lines of flow are radial and symmetrical.

§ 52. As a second example we shall calculate the resistance of an electrolyte which is of infinite extent in all directions, in which two spherical electrodes—the anode of radius a, the cathode of radius b—lie at a great distance R from each other. We may suppose the return connection from the cathode to the anode to be made by means of a wire which is carefully insulated from the electrolyte and which is so thin that the electric field is not appreciably disturbed by it (cf. end of § 25). Any electromotive force may be applied to it.

With these assumptions the potential ϕ in the electrolyte can be represented by the following particular solution of Laplace's differential equation:

$$\phi = \frac{A}{r_1} + \frac{B}{r_2} \quad . \quad . \quad . \quad . \quad . \quad (259)$$

where r_1 and r_2 denote the distances of the reference-point from centres of the spheres; A and B denote two constants which are determined by the values ϕ_1 and ϕ_2 of the potential function at the surfaces of the two electrodes. For on the first sphere we have $r_1 = a < < r_2$; hence $\phi = \frac{A}{a} = \phi_1$. At the second sphere $r_2 = b < < r_1$; hence $\phi = \frac{B}{b} = \phi_2$. From this it follows by (259):

$$\phi = \frac{a\phi_1}{r_1} + \frac{b\phi_2}{r_2} \quad . \quad . \quad . \quad . \quad . \quad (260)$$

On the other hand, (247) gives for the total intensity of current, if we choose in the one case the anode surface 1 ($\nu = r_1$) as the cross-section of the current and in the other the cathode surface 2 ($\nu = -r_2$):

$$J = -\kappa\int\frac{\partial\phi}{\partial r_1}d\sigma_1 = \kappa\int\frac{\partial\phi}{\partial r_2}d\sigma_2 \quad . \quad . \quad . \quad . \quad (261)$$

or, by (260), if we perform the integrations over the surfaces of the two spheres:

$$J = 4\pi\kappa a\phi_1 = -4\pi\kappa b\phi_2.$$

This gives for the total resistance of the infinitely extended electrolytic conductor:

$$W = \frac{\phi_1 - \phi_2}{J} = \frac{1}{4\pi\kappa}\left(\frac{1}{a} + \frac{1}{b}\right) \quad . \quad . \quad . \quad (262)$$

This result allows us to draw an interesting conclusion about the electric resistance of the earth. For it is obvious that the system of lines of flow in the above electrolyte are in no wise altered if we place an insulated horizontal plane of infinite extent through the centres of the two spherical surfaces and if we replace the upper half of the infinite space above the horizontal plane by an insulator (air) which can also be the receptacle of the return lead from the cathode to the anode (telegraph wire) with the electromotive force which it carries. The expression (260) for the potential function then remains unaltered. On the other hand, the integrals in (261), since the cross-section of the current is halved, also become reduced to half their values, so that we obtain for the resistance of the part of the earth limited by the horizontal surface between the two widely separated hemispherical electrodes of radii a and b:

$$W = \frac{1}{2\pi\kappa}\left(\frac{1}{a} + \frac{1}{b}\right) \quad . \quad . \quad . \quad . \quad . \quad (263)$$

in place of (262). This expression does not depend at all on the distance of the electrodes, but only on their size.

When this remarkable fact first became known, some physicists were led by it to assume that the earth's resistance has its seat exclusively at the surfaces of the electrodes, that is, it is to be regarded as a sort of superficial resistance (*Übergangswiderstand*), whereas the body of the earth itself has no resistance at all. As we have seen, this view is in no wise justified. The apparent paradox that the earth's resistance is independent of the

distance R between the electrodes is explained immediately if we reflect that the increase which the resistance acquires through the increased length of the current-filaments is exactly compensated by the decrease of resistance caused by the fact that the current-filaments penetrate more deeply into the body of the earth and hence produce an increase in the cross-section of the current.

§ 53. Finally, let us consider in greater detail the case where the system of conductors fills a space which is more than doubly connected or, as we say, has *branch-points*. We shall restrict ourselves to linear conductors. These then in general form an irregularly arranged network whose knots form the "branch-points." Each branch-point is the meeting-point of three or more conductors. The portions of conductors between each pair of neighbouring branch-points are called the "branches" of the system of conductors (or circuit). In every branch we may introduce a special electromotive force, which we shall suppose to be given. Then, as G. Kirchhoff has shown, there are two simple rules by which we can calculate the stationary current-strength, which will, of course, have a special value in each branch. Firstly, since the process is stationary the algebraic sum of all the quantities of electricity that flow into each branch-point in a definite time must be zero for each branch-point, that is:

$$\Sigma J = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (264)$$

Secondly, for every individual branch the relation (249a), where ϕ_1 and ϕ_n denote the values of the potential function ϕ at the initial point and the end point of the branch, must hold. Now if we combine any adjacent branches in continuous succession to form a closed circuit and add the corresponding relations (249a) together, the values of ϕ cancel in pairs and we get Kirchhoff's second rule:

$$\Sigma E = \Sigma J \, . \, w \, . \quad . \quad . \quad . \quad . \quad . \quad (265)$$

where the quantities E, J and w each relate to an individual branch of the closed circuit. In applying the formula we must take care that the positive direction of E and J corresponds to a definite clockwise or anti-clockwise motion round the circuit.

§ 54. Let us apply Kirchhoff's laws to the ordinary Wheatstone Bridge, which is the system of conductors shown in Fig. 7; we shall calculate the current-strengths for the arbitrarily given resistances w in the main circuit, w_1, w_2, w_3, w_4 in the four branches and w_0 in the bridge. Suppose an electromotive force E to be acting in the main current. The directions in which the quantities E and J are to be considered positive in the calculations are shown in the figure by arrows. From (264) we then get the relations :

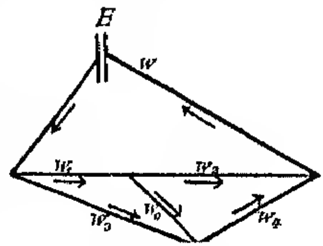

FIG. 7.

$$J = J_1 + J_3 = J_2 + J_4$$
$$J_1 = J_0 + J_2,\ J_3 + J_0 = J_4.$$

And from (265) we get the relations :

$$E = Jw + J_1w_1 + J_2w_2$$
$$E = Jw + J_3w_3 + J_4w_4$$
$$0 = J_1w_1 + J_0w_0 - J_3w_3$$
$$0 = J_2w_2 - J_4w_4 - J_0w_0$$

These equations determine the current-strengths in all the branches. Thus for the current in the main circuit we get :

$$J = \frac{\{w_0\Sigma w_1 + (w_1 + w_3)(w_2 + w_4)\}.E}{\Sigma w_1w_2w_3 + w(w_1+w_3)(w_2+w_4) + w_0(w_1+w_2)(w_3+w_4) + ww_0\Sigma w_1} \qquad (266)$$

where the sums Σ are to be performed over the four branches 1, 2, 3, 4. For the current in the bridge we have :

$$J_0 = \frac{(w_2w_3 - w_1w_4).J}{w_0\Sigma w_1 + (w_1 + w_3)(w_2 + w_4)} \quad . \qquad (267)$$

If there is to be no current in the bridge the simple condition:

$$w_1 : w_2 = w_3 : w_4 \quad . \quad . \quad . \quad . \quad . \quad (268)$$

must hold, as is clear from (267).

§ 55. Having seen in the preceding sections how the electric potential and the current-strength may be determined in a stationary field, we shall now assume the electrical quantities to be known and shall investigate how the magnetic field, which they call up, must be constituted. The relation (241) between the magnetic field-strength $\boldsymbol{H}$ and the current-density $\boldsymbol{J}$, in conjunction with the general relations (51) and (52), furnish the data.

To simplify the discussion without losing sight of the characteristic features, we shall in the sequel, unless the contrary is expressly stated, always assume the permeability μ to be of the same value in all substances, be they conductors or insulators. For then the field-strength $\boldsymbol{H}$ is continuous at all the boundary surfaces, on account of (11) in the case of the tangential components, and on account of (52) in the case of the normal component.

In this form the problem of calculating the magnetic field-strength $\boldsymbol{H}$ from the current-strength $\boldsymbol{J}$ is shown to be completely identical with the problem of calculating the velocity $\boldsymbol{q}$ of the liquid, assumed steady, from the vortical velocity $\boldsymbol{o}$ of an incompressible liquid (II, § 73), and we are in a position to make use of the laws which correspond in the present case to those which hold in hydrodynamics. From this we deduce the important result that the current-density $\boldsymbol{J}$ uniquely determines the magnetic field-strength $\boldsymbol{H}$ in the whole of space, inside and outside the currents.

Since by (241) curl $\boldsymbol{H} = 0$ outside the currents the magnetic field in the insulators has a potential ϕ. The question is whether this magnetic potential ϕ is one-valued and continuous. The condition for this is identical with that which requires that the integral:

$$\oint \boldsymbol{H}_x dx + \boldsymbol{H}_y dy + \boldsymbol{H}_z dz = A \quad . \quad . \quad . \quad . \quad (269)$$

should be zero for every closed curve that lies entirely within the insulators. Now it follows from the relation curl $\boldsymbol{H} = 0$ that the integral (269) retains its value when we alter the curve to an infinitesimal extent (II, § 69). Thus the question as to whether the magnetic potential is one-valued and continuous reduces to the question as to whether the closed curve which lies entirely within the insulators can be contracted continuously to zero. For if this can be done the constant value of (269) necessarily becomes zero. Since in the present case the insulators, like the conductors which are being traversed by currents, form a multiply connected space, the question which has been proposed must in general be answered in the negative. It is, in fact, not possible to contract to zero a curve which lies in an insulator and which encircles a conductor through which a current is passing.

If then, on account of the current passing through the conductor, the value of the integral is not equal to zero, this current will have a definite connection with the curve in question. This connection is, indeed, represented quite generally by a simple relationship which is based on a mathematical transformation of the equation (241), which is called Stokes's Theorem.

§ 56. To prove Stokes's Theorem we must calculate the change which the integral A undergoes when the closed curve of integration is altered to an infinitesimal degree for the general case when curl $\boldsymbol{H}$ is not equal to zero. We get for this:

$$\delta A = \oint \delta \boldsymbol{H}_x dx + \delta \boldsymbol{H}_y dy + \delta \boldsymbol{H}_z dz$$

$$+ \oint \boldsymbol{H}_x \delta dx + \boldsymbol{H}_y \delta dy + \boldsymbol{H}_z \delta dz \quad . \quad . \quad (270)$$

If we integrate by parts, using the formula

$$\oint \boldsymbol{H}_x \delta dx = -\oint d\boldsymbol{H}_x \delta x, \cdots$$

and take into account the fact that the field-components $\boldsymbol{H}_x$, $\boldsymbol{H}_y$, $\boldsymbol{H}_z$, depend only on x, y, z, we get :

$$\delta A = \oint \left(\frac{\partial \boldsymbol{H}_z}{\partial y} - \frac{\partial \boldsymbol{H}_y}{\partial z} \right) (\delta y dz - \delta z dy) + \ldots \quad . \quad (271)$$

The first factors after the integral sign are here the components of the vector curl $\boldsymbol{H}$, the second factors are the components of the vector product of $\delta \boldsymbol{r}$ and $d\boldsymbol{r}$, whose absolute value is equal to the surface area of the parallelogram formed by the displacement $\delta \boldsymbol{r}$ and the curve-element $d\boldsymbol{r}$, and whose direction is normal to the parallelogram, so that it forms a right-handed co-ordinate system with the directions $\delta \boldsymbol{r}$ and $d\boldsymbol{r}$. Now the parallelogram is nothing other than the area of surface $d\sigma$ which is swept out by the curve-element $d\boldsymbol{r}$, the normal to the area being ν. Hence we may write :

$$\delta A = \oint (\text{curl } \boldsymbol{H})_\nu \cdot d\sigma \quad . \quad . \quad . \quad . \quad (272)$$

where the integration is to be taken over all the elements $d\sigma$ of the infinitely narrow ring-shaped surface formed by the original and the displaced curves. The direction of ν is the sense of the rotation denoted by the direction of the curve-element after the displacement and by the opposite direction of the original curve-element (I, § 83).

If we now consider any finite surface bounded by a given closed curve and composed of elements $d\sigma$, we can regard this surface as made up entirely of infinitely narrow ring-shaped areas, the outermost of which is bounded by the given boundary-curve, whereas the innermost contracts into a single infinitely small area of surface. For each of these ring-shaped surfaces we form an equation on the model of (272) and obtain by adding together all these equations :

$$A = \oint \boldsymbol{H} \cdot d\boldsymbol{r} = \int (\text{curl } \boldsymbol{H})_\nu \cdot d\sigma \quad . \quad . \quad (273)$$

as the expression of Stokes's Law, since on the left-hand side the line-integrals cancel out in pairs except the

outermost, which is to be taken over the given boundary curve, and the innermost, which is equal to zero. The normal ν of the surface-element $d\sigma$ is to be taken in the direction which corresponds to the axis of the rotation defined by the integration over the curve-elements dr.

Since the integral on the left depends only on the boundary-curve, but not on the special position of the surface σ, the same holds for the integral on the right-hand side. This can also easily be seen directly if we consider that two different surfaces σ having the same boundary-curve together form a single closed surface, which marks off a definite volume of space, and that the surface-integral (273) over this closed surface is equal to the space-integral over the quantity div. curl $\boldsymbol{H}$, which vanishes identically. This can also be expressed by saying that the "total flux" of a vector which is free of divergence through any surface depends only on the boundary-curve, but not on the shape of the surface in any other way. If the surface is closed, the vector-flux is equal to zero. An example of a vector which is free of divergence in this way is the velocity of an incompressible liquid; another is the magnetic induction $\boldsymbol{B}$, for which we have already earlier established the corresponding relation in the form (53).

§ 57. Reverting to our electromagnetic problem, we now obtain, by substituting (241) in the form (273) of Stokes's Theorem, the expression:

$$\oint \boldsymbol{H} \cdot d\boldsymbol{r} = \frac{4\pi}{c} \int J_\nu d\sigma = \frac{4\pi}{c} \cdot J \quad . \quad . \quad . \quad (274)$$

or, expressed in words: *the integral of the magnetic work along a closed curve is equal to* $\frac{4\pi}{c}$ *times the quantity of electricity which flows per unit of time through any surface bounded by this curve:* moreover, this magnetic work is positive if the sense of rotation about the axis as denoted by the integration is in the direction of the current. This theorem is the foundation of electromagnetism and is

independent of all material constants. The fact that the value of J does not depend on the particular shape of the surface follows in accordance with the reasoning used at the conclusion of the preceding section from the fact that the vector of the current-density $\boldsymbol{J}$ is free from divergence. So long as the closed boundary-curve lies entirely in the insulator a change in its position in no way affects the value of J, and the magnetic work remains constant. If $J=0$ the curve can be gradually and continuously contracted to zero without leaving the insulator: the magnetic work is then also equal to zero. But if J differs from zero, then a moment will occur when the curve will enter into the conductor traversed by the current. As the contraction proceeds further the value of J decreases until it finally becomes equal to zero, when the curve reduces to a point. In this way, by extending the physical significance of the theorems so far proved, we see how they gain in content and clearness.

§ 58. Let us now determine the magnetic field produced by a current for a special case, namely for a *linear* current which is present in any homogeneous insulator; let the current be of given finite intensity $J>0$. Then, as we know, the magnetic field-strength $\boldsymbol{H}$ is everywhere uniquely defined and has the potential ϕ in the insulator. But this potential is not uniform and continuous. For the magnetic work along a closed curve which encircles the current circuit once is not equal to zero, but has a constant finite value defined by (274). Thus we have the choice either of assuming the magnetic potential ϕ to be continuous—then it is many-valued—or to assume that it is one-valued, and then it is discontinuous (cf. II, § 69 *et seq.*). Since we wish to restrict our calculation to one-valued quantities, we decide in favour of the latter alternative, but on *each* curve which envelops the current-circuit we must then fix on some ideal point A in passing through which the potential ϕ experiences a sudden constant change which is given by (274). We combine all these ideal points A into an ideal surface bounded by

the circuit, and this surface must be traversed if a curve is to be described around the current. The ideal surface of discontinuity introduced in this way has no physical significance. It only serves to enable us to calculate uniquely the potential function from the field-intensity in virtue of the convention that the line connecting two points along which the magnetic work $\boldsymbol{H} \cdot d\boldsymbol{r}$ is integrated must not pass through the surface of discontinuity if we wish to calculate the potential difference between the two points. This, in fact, converts the doubly-connected space which is occupied by the insulator into a singly-connected space, and all the closed curves that can be drawn in it can be continuously contracted to zero.

With this convention we can now set up all the properties of the magnetic potential function ϕ. ϕ is one-valued and finite inside the whole insulator. At infinity ϕ is zero, since we assume that the circuit or current-curve lies entirely in finite regions. At the ideal surface of discontinuity bounded by the circuit ϕ jumps by the amount $\frac{4\pi J}{c}$ in the positive sense if the reference-point passes through the surface of discontinuity in the direction of the axis of rotation corresponding to the direction of flow. This relationship between the directions is shown graphically in Fig. 8. The continuous curve represents the circuit, imagined horizontal and seen from above. By (274) the magnetic work along the dotted vertical curve which encircles the circuit is positive for the sense of integration denoted by the arrow. Hence the magnetic potential function ϕ along this curve gradually decreases until it jumps back to its original greater value, when its reference-point passes through the point A of the ideal surface of discontinuity bounded by the circuit.

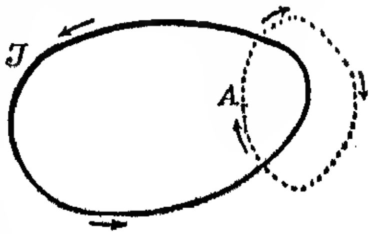

Fig. 8.

The first derivatives of the potential function, that is, the components of the field-strength H, are everywhere continuous except in the immediate neighbourhood of the circuit traversed by the current, where they become infinitely great, since the closed integral (274) retains a finite value even when the length of the circuit decreases to an indefinitely small amount. Finally the potential function satisfies Laplace's equation $\Delta\phi = 0$ everywhere in the insulator.

The function ϕ which is uniquely determined by all these conditions is, according to the general theorem derived in § 29, identical with the following potential function of a magnetic double-layer (or shell)[1] situated on the surface of discontinuity:

$$\phi = g \cdot \int \frac{\partial \frac{1}{r}}{\partial \nu} d\sigma$$

whose moment is calculated from the given potential jump by (151) and comes out as:

$$g = \frac{J}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (275)$$

and whose normal ν forms the axis of the rotation indicated by the direction of flow of the current. The potential function:

$$\phi = \frac{J}{c} \cdot \int \frac{\partial \frac{1}{r}}{\partial \nu} d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (276)$$

can be seen from the reasoning given in § 28 to possess all the properties here demanded of it. In particular, its first differential coefficients or the components of the magnetic intensity of field are continuous throughout and depend only on the boundary-curve or shape of the circuit, but in no other way on the position of the surface σ.

[1] To preserve the analogy with electrostatics we shall use the term double-layer instead of the term "shell," which is perhaps more familiar to English readers.

Through equation (276) all the magnetic actions of a linear current are reduced to those of a definite magnetic double-layer. But we must here add an important remark. Equation (276) is independent of all material constants, in particular also of the permeability of the insulator in which the current happens to be. Hence the value $\frac{J}{c}$ of the moment of the double-layer which occasions the potential jump does not refer to the real magnetic charge, but to the free magnetic charge just as, by (88), in the case of a simple electric or magnetic surface-layer the discontinuity of the normal component of the field-intensity is determined not by the real, but by the free charge. Hence if we inquire into the true magnetic moment of charge of the double-layer which is equivalent to the current, then, on account of the relation (92) the value of the free charge must yet be multiplied by the permeability μ. The result so obtained may be formulated as follows. *The magnetic field generated by a stationary linear current in an insulator of permeability μ is identical with that of a double-layer bounded by the circuit curve and having the free charge-moment $\frac{J}{c}$ or the true charge-moment $\frac{\mu J}{c}$.*

The distinction we have just made becomes of importance when we deal with interactions of currents, as we shall see later. Here, where we are concerned only with the magnetic field produced by a current, it is sufficient to consider the free charge.

§ 50. Since we have learned how to measure all magnetic quantities in absolute Gaussian units, the above theorem, or equation (276), gives us a means of measuring the quotient $\frac{J}{c}$ directly. This is the first time that we encounter the critical velocity c in a relationship which can be applied to measurements. So long as the value of c is not known, the value of the current strength J in

Gaussian units cannot be derived from magnetic measurements alone. Now, the quotient $\frac{J}{c}$ denotes the current-strength measured in Maxwell's electromagnetic units (§ 7) (cf. also the table of units at the end of the book). Hence the magnetic measurements give the latter quantity directly, and that is the reason why the practical system of units has been linked up with Maxwell's electromagnetic system and not with Gauss's electrostatic system.

By (150) and (275) a circular current of radius R produces at its centre the magnetic intensity of field:

$$|H| = \frac{J}{c} \cdot \frac{2\pi}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (277)$$

If we set $R = 1$ and consider that the current then has the length 2π, we may also say: in Maxwell's electromagnetic system the unit of current strength is that which when it flows in a circular arc of unit radius and unit length acts with unit force on a unit magnetic pole situated at the centre of the arc.

§ 60. To find the value of the critical velocity c we must combine the magnetically measured quantity $\frac{J}{c}$ with the electrically measured quantity J. This can be done by means of a method first used by R. Kohlrausch and W. Weber. Its principle consists in conducting the discharge of an electric condenser through a circular wire at whose centre a small magnetic needle is situated. This conducting circuit is placed vertically in the plane of the magnetic meridian. The position of rest of the needle then lies in the plane of the needle. If the current is now discharged through the conducting system the needle experiences a turning moment and begins to oscillate. The values of the quantities involved can then be deduced in the well-known way. It is true that we are not, of course, dealing with a stationary current, but, on the contrary, with one that lasts only for a very short time and is consequently very variable. But we shall nevertheless

not introduce an appreciable error by assuming that there is no accumulation of electric charge at any point of the wire in the circuit; the result is that the current-strength J has the same value at all points of the circuit at a definite time t and hence may be considered to be a function of the time t alone, which is zero initially and then rapidly increases and then again decreases to zero. Further, we shall assume that the magnetic field produced by the current J is constituted at every moment of time t as if the current were stationary. These simplifications anticipate for this special case those which occur for "quasi-stationary" events which will be discussed in their proper place in Part Three of the present volume. We shall there also deal with their range of application, which is very wide in practice.

In the light of these remarks we see that for any arbitrary time t the field-strength H caused by the discharge current J and acting on the magnetic needle is given by (277). Hence at this particular moment the needle experiences a turning-moment (234). Without introducing an appreciable error we can assume the discharge current to pass so quickly that the discharge is fully completed before the needle has been appreciably disturbed from its position of equilibrium. In other words the mechanical action of the current may be regarded as a momentary action (ballistic galvanometer) or as an impulse (I, § 130). The value of the vector-product (234) is then simply equal to the product of the moment M of the needle and the field-strength H. Thus the equation of motion of the needle is, by I (409*b*):

$$K \cdot \frac{d\omega}{dt} = M \cdot \frac{J}{c} \cdot \frac{2\pi}{R}$$

where we denote the moment of inertia by K and the angular velocity of the needle by ω.

We integrate this equation with respect to the time from $t = 0$, the moment at which the discharge begins, over an interval of time $t = \tau$ which is chosen so great

that the discharge is fully completed within it, but at the same time so small that the needle still lies in the plane of the magnetic meridian. We then get, since $\omega = 0$ for $t = 0$:

$$K \cdot \omega_\tau = \frac{M}{c} \cdot \frac{2\pi}{R} \cdot \int_0^\tau J dt \quad . \quad . \quad . \quad . \quad (278)$$

Here ω_τ is the angular velocity with which the magnetic needle begins its observable oscillations about its position of equilibrium, after which it is only under the influence of the horizontal component of the earth's magnetic field (Gauss's vibration experiment, § 45). Since the laws governing these oscillations are well known, it is possible to calculate ω_τ, say by measuring the amplitude of vibration, and also M and K. Further:

$$\int_0^\tau J dt = e$$

which is the total quantity of electricity that has flowed through the wire in a definite direction and is independent of the special form of the time function J; it is equal to the electric charge of the condenser, which can also be measured in Gaussian units, say with Thomson's electrometer (§ 43).

All the quantities are now known which enable us to calculate the only unknown in the relation (278), the critical velocity c. Results agree in giving for this:

$$c = 3 \cdot 10^{10} \text{ cm. sec.}^{-1} \quad . \quad . \quad . \quad . \quad . \quad (279)$$

and from now on we may regard this quantity as known for all later calculations.

The method above described, by which c was first determined, is a particularly beautiful example not only of the art of accurate measurement, but also of the visionary power of theory which achieves its greatest successes in complicated cases by teaching us how to sift out and discard unessentials, so that we are left with the essentials.

In carrying out the measurement in practice the dis-

charge current is led round the magnetic needle not in one but in many turns of wire (coils, solenoid), since, by (274), this makes the intensity of field correspondingly stronger.

§ 61. The knowledge of the value of the critical velocity establishes a bridge once and for all between the different systems of measurement. Owing to the enormously great numerical value of c a current intensity that has been physically measured in the electromagnetic system of units is represented by a much smaller number $\left(\frac{J}{c}\right)$ than in the Gaussian electrostatic system. The reverse is the case with the field-intensity E or the electromotive force E. For in the case of these quantities the relations (16) hold, from which it is seen that the field-strengths and potentials of a definite electric field in the electromagnetic system are expressed by a number which is c times greater than in the Gaussian system.

Since the conductivity or the resistance is, by (251), given by dividing the electromotive force by the strength of current, the contrast between the two systems, to which we have just referred, becomes still more pronounced. Thus for mercury at 0° C. the specific conductivity in the Gaussian system is:

$$\kappa = 9{\cdot}567 \,.\, 10^{15}\ \text{sec.}^{-1} \quad . \quad . \quad . \qquad (280)$$

whereas in the electromagnetic system it is:

$$\frac{\kappa}{c^2} = 1{\cdot}063 \,.\, 10^{-5}\ \text{cm.}^{-2}\ \text{sec.} \quad . \quad . \quad . \qquad (281)$$

In the *practical* system of units the inconvenient powers of ten that occur in the absolute systems are avoided as far as possible. Hence it has been agreed to take as the starting point of the practical system the convention that a resistance which has the value 10^9 cm. sec.$^{-1}$ in electromagnetic measure is to be called 1 ohm, and that an electromotive force or potential difference that has the value 10^8 grm.$^{\frac{1}{2}}$ cm.$^{\frac{3}{2}}$ sec.$^{-2}$ in the

electromagnetic system is to be called 1 volt. The practical unit of current, the ampère, then results from the equation (251), which represents Ohm's law; it amounts to $\frac{1}{10}$ of the unit of current in the absolute electromagnetic system.

If we were to let matters rest at these definitions the values of the practical units would depend on the accuracy with which we could realize the absolute units, and this would lead to very unsatisfactory practical results. It has therefore been further agreed to adopt the course followed earlier in the case of the unit of length (I, § 3) by introducing in addition to the above theoretical definitions certain legal or international definitions which agree very closely with them and have the advantage that they can be reproduced much more accurately. Thus the international ohm is the resistance of a column of mercury of cross-section one square millimetre and 106·3 cms. long. By (249) this resistance when expressed in the electromagnetic system is:

$$\frac{c^2}{\kappa}\cdot\frac{l}{q}=\frac{c^2}{\kappa}\cdot\frac{106\cdot 3}{10^{-2}}=\frac{c^2}{\kappa}\cdot 10630$$

which, by (281), is equal to 10^9.

Further, the international ampère is that current which deposits 0·001118 grm. of silver in one second (cf. § 66). If one ampère flows through a circular arc of 1 cm. radius it produces, by (277) a magnetic intensity of field equal to $\frac{2\pi}{10}$, that is, 0·628 gauss.

The value of the international volt then comes out of equation (251), which is Ohm's law. The electromotive forces of the most commonly used galvanic cells lie between 1 and 2 volts. In particular, the electromotive forces of the standard Weston cell (cadmium) is 1·01830 international volts at 20° C.

The practical units so far mentioned also fix the values of the practical units of quantity of electricity and electric capacity; thus the coulomb is the quantity of

electricity that is supplied by 1 ampère in 1 second, and the farad is the capacity which 1 coulomb raises to a potential of 1 volt.

The millionth part of these quantities bears the prefix micro-, and quantities which are a million times as great bear the prefix mega, thus we speak of micro-farads, micro-volts and so forth.

The capacity of an electric condenser with air as the dielectric whose surface is 100 square cms. and distance between the plates is 1 cm. amounts by (70) to:

$$\frac{100}{4\pi} \cdot \frac{1}{c^2} \cdot 10^9 \,.\, 10^6 = 8{\cdot}84 \,.\, 10^{-6} \text{ micro-farad.}$$

§ 62. We now turn from the magnetic actions of linear currents to the general case of *space* currents. This brings us back to the problem formulated in § 55, which we shall now endeavour to solve directly by calculating the magnetic intensity $\boldsymbol{H}$ at any point of space, inside or outside the currents, from the given current-densities $\boldsymbol{J}$. In doing so we shall follow the same method as in II, § 74, except that here we shall use vector methods.

We first set up as the general integral of the differential equation (51), which is valid everywhere, the following relation:

$$\boldsymbol{B} = \text{curl}\, \boldsymbol{A} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (282)$$

where the vector $\boldsymbol{A}$ can be a purely arbitrary function of position. Since $\boldsymbol{B}$ is everywhere continuous, we shall also assume $\boldsymbol{A}$ and its first differential coefficients to be continuous. Besides this, however, we may also impose a further restriction on the vector $\boldsymbol{A}$ without influencing the value of $\boldsymbol{B}$ in any way. For if in place of $\boldsymbol{A}$ we write $\boldsymbol{A} + \text{grad}\, \psi$, then $\boldsymbol{B}$ clearly retains its value unchanged, however, ψ may be constituted. We use this circumstance to give the vector $\boldsymbol{A}$ the further property:

$$\text{div}\, \boldsymbol{A} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (283)$$

This can always be realized by appropriately disposing of

the scalar function ψ and in particular of the value of $\Delta\psi$. From (282) it then follows that:

$$\boldsymbol{H} = \frac{1}{\mu} \operatorname{curl} \boldsymbol{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (284)$$

and by substitution in (241) we obtain the following equation to determine $\boldsymbol{A}$:

$$\operatorname{curl} \operatorname{curl} \boldsymbol{A} = \frac{4\pi\mu}{c} \cdot \boldsymbol{J} \quad . \quad . \quad . \quad . \quad . \quad (285)$$

Now by performing the calculation for a particular component it is easy to verify the identity:

$$\operatorname{curl} \operatorname{curl} \boldsymbol{A} = \operatorname{grad} \operatorname{div} \boldsymbol{A} - \Delta \boldsymbol{A} \quad . \quad . \quad (286)$$

and hence by taking into account (283):

$$\operatorname{curl} \operatorname{curl} \boldsymbol{A} = -\Delta \boldsymbol{A} \quad . \quad . \quad . \quad . \quad . \quad (287)$$

and from (285):

$$\Delta \boldsymbol{A} = -\frac{4\pi\mu}{c} \cdot \boldsymbol{J} \quad . \quad . \quad . \quad . \quad . \quad (288)$$

This is Poisson's equation, not for a scalar, however, but for a vector. Hence the integral of this differential equation becomes:

$$\boldsymbol{A} = \frac{\mu}{c} \int \frac{\boldsymbol{J}'}{r} d\tau' \quad . \quad . \quad . \quad . \quad . \quad . \quad (289)$$

which is called "vector potential," to distinguish it from scalar potential. Here r denotes the positive distance of the volume-element $d\tau'$, which is filled with the current-density $\boldsymbol{J}'$, from the reference-point. Like all vector equations, the equation (289) represents three equations, one for each of the three components of $\boldsymbol{A}$ and $\boldsymbol{J}'$.

Substituting this equation in (284) we get for the required magnetic intensity:

$$\boldsymbol{H} = \frac{1}{c} \int \left[\operatorname{grad} \frac{1}{r}, \boldsymbol{J}' \right] d\tau' . \quad . \quad . \quad . \quad . \quad (290)$$

that is:

$$\boldsymbol{H} = \frac{1}{c} \int \frac{d\tau'}{r^2} \cdot \left[\boldsymbol{J}', \frac{\boldsymbol{r}}{r} \right] \quad . \quad . \quad . \quad . \quad . \quad (291)$$

where the vector $\boldsymbol{r}$ is in the direction leading from the volume-element $d\tau'$ to the reference-point. Cf. II (304).

Instead of the current-density $\boldsymbol{J}$ we can also introduce the current-strength J of an infinitely thin current-filament of cross-section q. Let us take $d\tau$ as representing an infinitely short piece of such a current element of length ds. Then, since $d\tau = q \,.\, ds$ and $J = |\boldsymbol{J}| \,.\, q$, we have :

$$|\boldsymbol{J}|\, d\tau = J ds, \qquad (291a)$$

that is, the vector :

$$\boldsymbol{J}\, d\tau = J d\boldsymbol{r} \qquad (292)$$

and we get :

$$\boldsymbol{H} = \frac{1}{c}\int \frac{J}{r^2} \cdot \left[d\boldsymbol{r}, \frac{\boldsymbol{r}}{r} \right] \qquad (293)$$

According to this the magnetic field-strength $\boldsymbol{H}$ at the reference-point shows itself to be the resultant of an infinite number of small intensities of field, each of which is due to a definite current-element $J \,.\, d\boldsymbol{r}$ and may be regarded as the field-strength produced at the reference-point by this current-element. Its magnitude and direction are determined by the vector :

$$\frac{J}{c} \cdot \frac{1}{r^2} \cdot \left[d\boldsymbol{r}, \frac{\boldsymbol{r}}{r} \right] \qquad (294)$$

Hence its magnitude is $\dfrac{J ds}{c} \cdot \dfrac{1}{r^2} \cdot \sin\,(d\boldsymbol{r}, \boldsymbol{r})$ and its direction is perpendicular to the plane through ds and r and is such that to a swimmer who is swimming in the element ds with the current and is looking at the reference-point the magnetic force will act towards the left (Biot and Savart's Law, Ampère's Swimming Rule, Fleming's Left-Hand Law).

It is clear that the above results contain those obtained earlier for linear currents as special cases. For example, we directly obtain from (293) the well-known expression (277) for the magnetic intensity of field which a circular current of radius R produces at its centre.

The relation (291) also gives us, of course, the magnetic field-strength $\boldsymbol{H}$ in the interior of a conductor traversed by a current. In this case $\boldsymbol{J}$ differs from zero at the

reference-point, but there is no direct relationship between the quantities $\boldsymbol{H}$ and $\boldsymbol{J}$. In particular, it must not be imagined that $\boldsymbol{H}$ must be perpendicular to $\boldsymbol{J}$. It may equally well happen that the magnetic field-strength is in the same direction as the electric current.

§ 63. Since the magnetic state at any point is determined by the magnetic field-strength at that point, a magnetic field produced by a current is of precisely the same nature as a field produced by a magnet, and can in no way be distinguished from it physically. From this it follows that we may imagine any statical magnetic field which exists in nature to be produced either by magnets at rest or by stationary currents. For if stationary currents happen really to be in question we can imagine the whole flow of electricity to be composed only of infinitely thin, that is, linear, current-filaments, and we can then apply to each of them the law found in § 58, but we must take care that the surface of the magnetic double-layer is not allowed to pass through the reference-point, and secondly that the magnetic action of an infinitely thin current-filament at a reference-point in its interior can be neglected entirely when the current-density is finite.

But if permanent magnets are actually present, they form an infinite system of infinitesimal elementary magnets. For by applying the fundamental law of § 58 in the reverse direction every elementary magnet of magnetic moment $\boldsymbol{M}$ can be replaced by a surface-element σ of arbitrary shape traversed by a linear current J, its normal coinciding with the axis of the magnet, while:

$$\frac{\mu J}{c}\sigma = |\boldsymbol{M}| \quad . \quad . \quad . \quad . \quad . \quad . \quad (295)$$

All these infinitesimal currents taken together, called Ampère's molecular currents, produce the same field as the magnets. From the point of view of the general theory there can be no doubt as to which of these opposing views, the magnetic and the electric, is to be given the

preference. For since, on the one hand, the desire to unify our knowledge of physical nature impels us to replace the dualism between magnetism and electricity by a unitary view, on the other hand a reduction of electricity to terms of magnetism cannot come into question on account of the impossibility of interpreting the concept of quantity of electricity (real, not free), so there is nothing left but to regard electricity as the primary concept and magnetism as the derived concept. The hypothesis of Ampère's molecular currents has, in fact, led to consequences which have essentially been well confirmed throughout.

We shall now calculate the vector-potential A produced at any point in space at a distance r from a given elementary magnet of permanent moment M. We may make use of the following equation, which follows from (289) and (292):

$$A = \frac{\mu}{c} \cdot J \cdot \int \frac{dr}{r} . \qquad (296)$$

where the integration is to be taken over all the elements dr of the boundary-curve of the infinitely small surface σ. We can choose as the form of σ either a circle or, more conveniently, a rectangle, since the whole integration then reduces to a summation over the four sides of the rectangle. Since dr is a vector, we have to calculate three integrals which result from the above when we replace dr in it by its three components. Taking (295) into account, we then arrive at the required expression for A.

Instead of carrying out these elementary processes of calculation, we can also conveniently obtain the expression for A in the following way. The potential of the given elementary magnet is:

$$\phi = -\frac{M}{\mu} \cdot \operatorname{grad} \frac{1}{r} \qquad (297)$$

(Cf. the potential outside a sphere magnetized in the direction of the z-axis for $\mu_0 = \mu$. The case $\mu \neq \mu_0$ will be discussed in the next section.) We then have, by (282),

the following relations between the field-strength H, the induction B and the vector-potential :

$$\operatorname{curl} A = B = \mu H = \operatorname{grad}\left(M \cdot \operatorname{grad}\frac{1}{r}\right) \quad . \quad (298)$$

Since M does not depend on the position of the reference-point, we have identically :

$$\operatorname{grad}\left(M \cdot \operatorname{grad}\frac{1}{r}\right) = M \cdot \Delta\frac{1}{r} + \operatorname{curl}\left[\operatorname{grad}\frac{1}{r}, M\right]. \quad (299)$$

and by (298), since $\Delta\frac{1}{r} = 0$:

$$\operatorname{curl} A = \operatorname{curl}\left[\operatorname{grad}\frac{1}{r}, M\right]$$

In view of the divergence condition (283) this leads uniquely to the expression of the vector-potential of an elementary magnet of moment M :

$$A = \left[\operatorname{grad}\frac{1}{r}, M\right] \quad . \quad . \quad . \quad . \quad (300)$$

and in the same way the magnetic field of any arbitrary finite magnet can be reduced to the vector-potential of Ampère molecular currents.

Although from the point of view of the general theory we should now, in the light of our above remarks, reduce all magnetic actions to the vector-potential, yet in many cases, particularly in those involving stational fields, practical reasons lead us to prefer the scalar potential ϕ which is easy to manipulate mathematically, but we must bear in mind that the magnetic scalar potential is to be regarded only as a convenient auxiliary mathematical quantity.

§ 64. In the electromagnetic deductions we have made so far, from § 55 onwards, we have for simplicity always assumed that the permeability μ has the same value in all substances. If we now drop this restriction, the normal component of the field-strength H and that of the tangential component of the induction B are no longer

continuous. Then to solve the problem of finding the magnetic field for a given stationary current $\boldsymbol{J}$ we can proceed in one of two ways. We can obtain the result by using the vector-potential alone, so long as we take into account besides $\boldsymbol{J}$ the Ampère molecular currents corresponding to the temporary magnetization. Or we can more conveniently resolve the magnetic field-strength into two parts, one of which is due to the current $\boldsymbol{J}$ alone and which is determined by the vector-potential $\boldsymbol{A}$ of this current, according to (280), and the other has a scalar potential ϕ:

$$\boldsymbol{H} = \frac{1}{\mu} \operatorname{curl} \boldsymbol{A} - \operatorname{grad} \phi . \quad . \quad . \quad . \quad (301)$$

If $\Delta\phi$ is taken equal to zero, then this assumption satisfies both (241) and (51), while the discontinuities in $\boldsymbol{H}_\nu$ at the boundary surfaces, which are due to the variable nature of μ, serve to define ϕ more closely.

CHAPTER VI

MOLECULAR AND PONDEROMOTIVE ACTIONS IN THE STATIONARY FIELD

§ 65. By (20) in every volume-element $d\tau$ traversed by electricity electrical energy of amount:

$$dt \cdot d\tau \kappa \mathbf{E}^2 = dt d\tau \mathbf{J}\mathbf{E} = dt d\tau \frac{\mathbf{J}^2}{\kappa} \quad . \quad . \quad . \quad (302)$$

is transformed into mechanical (Joule) heat during the interval of time dt. This gives for the amount of heat developed by a current J in a homogeneous linear conductor of cross-section q and having the element and of length dl between the cross-sections 1 and 2, since $J = \mathbf{J} \cdot q$ and $d\tau = q \cdot dl$:

$$dt \cdot J^2 \cdot \frac{1}{\kappa}\int_1^2 \frac{dl}{q}$$

or, by (249):

$$dt \cdot J^2 \cdot w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (303)$$

If the current passes through several conductors in succession, then by addition we get for the total heat generated:

$$dt \cdot J^2 \cdot \Sigma w \quad . \quad . \quad . \quad . \quad . \quad . \quad (304)$$

This expression also holds when the current-strength J varies in an arbitrary manner with the time.

For a stationary current we can also write for (304), by (250):

$$dt \cdot J \cdot (\phi_1 - \phi_n + \Sigma E) \quad . \quad . \quad . \quad . \quad (305)$$

where ϕ_1 and ϕ_n denote the values of the electric potential at the initial and the final cross-section of the system of conductors in question.

If the conductors are extended in space, then we get for the heat generated between an equi-potential surface

$\phi = \phi_1$ in the conductor 1 and an equi-potential surface $\phi = \phi_n$ in the conductor n, by summing up (305) over all the infinitely thin current-filaments that lie adjacent to each other, the same expression (305) except that now J denotes the *total* current strength. Using (253) we may also write for this :

$$dt \, . \, J^2 \, . \, W \quad . \quad . \quad . \quad . \quad . \quad . \quad (306)$$

By (305) the heat generated in the whole of the closed current system is, for $\phi_1 = \phi_n$:

$$dt \, . \, J \, . \, \Sigma E \quad . \quad . \quad . \quad . \quad . \quad . \quad (307)$$

where ΣE now denotes the total electromotive force in the closed surface.

In technical electricity the product $J \, . \, \Sigma E$ is called the "activity" or "power" of the electromotive force ΣE. The unit of power is called the volt-ampère or the watt. A kilowatt = 1000 watts, is equivalent to about 1·36 horse-power. The amount of energy delivered by 1 watt in 1 second is called the watt-second, and is equal to 10^7 ergs : the watt-second is also called a joule.

§ 60. The next question is : what is the source of energy that delivers this heat, which is always generated in a positive sense, in a stationary current at rest? It is obvious that only *molecular*, that is, thermal or chemical processes, come into question here. Moreover, if the conductors are all homogeneous and are kept at a uniform temperature, these processes occur exclusively at the boundary surfaces of the conductors. Since the electromagnetic field is invariable in time, and hence also has an invariable amount of energy, the electric current only plays the part of an intermediate carrier, and the total molecular energy used is equal to the total amount (307) of the Joule heat. Consequently the molecular energy *generated* at the boundary surfaces, namely :

$$- \, dt \, . \, J \, . \, \Sigma E \quad . \quad . \quad . \quad . \quad . \quad . \quad (308)$$

is proportional to the current-strength and the total electromotive force which acts at the boundary surfaces.

But this result holds only for the current circuit as a whole. It would be premature and, in general, incorrect to deduce further from this that the molecular energy generated by the current J at an individual boundary surface by the electromotive force E is given by the product $J \cdot E$. For since the flux through a single boundary surface does not form a closed system (I, § 120), the energy-principle does not allow us to make assertions about this molecular energy unless the external actions are taken into consideration; in the present case these actions are conditioned by the electricity that flows into and out of the boundary surface.

In metallic conductors the molecular energy (308) that arises at the boundary surfaces is restricted to the generation of heat (Peltier heat); in electrolytic conductors this is supplemented by the chemical energy connected with the decomposition of electrolytes. Here the general empirical law of Faraday holds, which states that the quantity of an electrolyte decomposed owing to the passage of a current depends solely on the total quantity of electricity that has flowed though it in a definite direction, irrespective of the time, the current-strength, the current-density, the electromotive force and so forth; thus it depends only on:

$$\int J dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (309)$$

and in the case of different electrolytes the quantities of any arbitrary substances deposited for a definite value of (309) are in the ratio of their equivalent weights, that is, in the ratio of the weights in which the substances combine chemically with one another. If a denotes the equivalent weight of a substance referred to oxygen = 16 grammes, then a current J in Gaussian units or $\frac{J}{c}$ in electromagnetic units, that is $\frac{10J}{c}$ ampères, deposits:

$$1{\cdot}0363 \cdot 10^{-5} \cdot a \cdot \frac{10J}{c} \cdot dt \text{ grammes} \quad . \quad . \quad (310)$$

of the substance in dt seconds. The coefficient $1{\cdot}0363 \,.\, 10^{-5} \,.\, a$ is also called the electrochemical equivalent of the substance.

Conversely, to decompose an equivalent weight a grammes of a substance we must pass through it the quantity of electricity $\frac{a}{1{\cdot}0363 \,.\, 10^{-4}} = 2{\cdot}895 \,.\, 10^{14}$ units in Gaussian measure or 9650 electromagnetic units or 96,500 coulombs.

For silver, $a = 107{\cdot}88$ and hence its electrochemical equivalent is:

$$1{\cdot}0363 \,.\, 10^{-5} \,.\, 107{\cdot}88 = 0{\cdot}001118 \text{ grm.}$$

(cf. § 61).

§ 67. To visualise the energy processes in a stationary galvanic circuit use has often been made of the analogy of a liquid which is slowly forced in a stationary current through a rigid pipe-system which exerts friction on the liquid. The heat arising from the friction would then correspond to the Joule heat, the pressure gradient to the potential gradient; and the amount of molecular energy used up at the boundary of two conductors would be represented by the action of a special mechanism which pumps the liquid at the junction of two pipes to a higher pressure.

This analogy, however, is fundamentally at fault in one respect, in that it gives a totally insufficient idea of the direction in which the energy is transported. For in the case of liquid flow the energy, being the mechanical work performed by the pressure acting on the liquid, flows in the direction of the lines of flow, whereas in the case of the galvanic current we know from Poynting's Theorem (9) that the direction of the flow of energy is perpendicular to the direction of the field and hence also to the direction of the current. Accordingly, every cylindrical current-filament in the interior of a conductor receives its Joule heat not through the two end surfaces, but through the surface of the cylinder itself, and the

same holds for the whole conductor. In this way we arrive at the conclusion—which seems very strange at first sight—that the Joule heat of a conductor is furnished, not by the surrounding conductors, but by the surrounding insulator, through whose boundary surface the energy enters the conductor from all directions transversely to the lines of flow. Since the state is stationary, the insulator must now of course have the energy which it gives up replaced; reflection shows that this occurs at the points where the insulator touches the surfaces of separation of any two conductors. For by § 27 the electric intensity of field here has very great singular values in the insulator. This is of course not to be understood as meaning that at every single point of contact of this kind the energy flows out into the insulator, since the sign of the electric intensity of field in the insulator changes with that of the contact potential E, while, on the other hand, the direction of the magnetic intensity of field is given by the direction of the current and is not influenced by the sign of E. But the algebraic sum of the streams of energy from all the points of contact signifies the transference of energy to the insulator and its amount is represented by (307), as can be shown by a detailed calculation, which we shall, however, omit here.

However strange and unnecessary this method of balancing the energy may at first sight appear, it nevertheless corresponds much more closely with the actual circumstances than that given in the analogy of a flow of liquid as described above. For so soon as we pass from stationary to variable galvanic current-strengths phenomena manifest themselves which prove directly that the insulator in no wise plays the passive part in current flow that is attributed to it in the mechanical picture of the rigid system of pipes completely sealed from their surroundings. The mere fact that in the case of stationary flow within an insulator there is an electro-magnetic field and hence also magnetic energy present shows that these factors cannot entirely fail to participate in the process

of flow. We know as an actual fact that when a galvanic current is generated the energy to be transferred to the insulator, which was originally neutral electromagnetically when no current was flowing in the conductors, has to be supplied by the system of conductors, if the flow is to be stationary; that is, this energy has to be supplied by the molecular energy at the surfaces of contact, and when the current is interrupted this amount of energy either returns to the system of conductors or, if the interruption is very abrupt, dissipates itself outwards into infinite space. We shall learn more about this below, from § 76 onwards.

§ 68. We shall now pass on from molecular phenomena in a stationary field to the consideration of *ponderomotive* actions or the mechanical forces which must be applied to any stationary system of freely movable magnets and conductors traversed by currents to keep them permanently at rest.

We can group these actions under three headings: firstly the mutual actions between the magnets (magnetic actions), secondly the interactions between the magnets on the one hand and the conductors carrying the currents on the other (electro-magnetic actions), and thirdly the interactions between the conductors among themselves (electrodynamic actions).

Concerning the *magnetic* ponderomotive actions we have learned the law governing them completely in § 45, and, indeed, in two different forms, each of which has its peculiar advantages, namely the integral or potential law and the differential or force law. According to the potential law the work performed during an infinitely small displacement of the magnets by the mechanical forces acting between them is equal to the simultaneous decrease of the magnetic potential, and the magnetic potential is identical with the magnetic energy U. According to the force law the resultant of the mechanical forces acting on any part of a magnet is obtained by compounding the magnetic pressures (230) acting on the surface of the part in question. Since the meaning of

the tensor of magnetic pressure is independent of whether the field is due to magnets or currents or whether the space enclosed by the surface encloses a magnet or a conductor conveying a current, this form of the differential law is suitable for direct application also to electromagnetic and electrodynamic effects, but it is often very difficult to form a picture with its aid. For this reason we shall always start out in the following sections from the integral law.

§ 69. We have next to deal with *electromagnetic actions.* Let us imagine any permanent magnet and any linear conductor carrying a current, both freely movable, acting on each other, then, on account of the magnetic field produced by the current according to § 58, the current exerts a mechanical action on the magnet, and conversely, by the principle of action and reaction (Newton's third law) the magnet will exert exactly the opposite action on the conductor carrying the current. The mechanical work performed during any displacement of the magnet and the current will, by § 45, be equal to the simultaneous decrease of the potential of the magnet with respect to the current or of the electromagnetic potential V, if we use this term to denote the potential of the field $\boldsymbol{H}$ produced by the magnet with respect to the magnetic double-layer which is equivalent to the current $\boldsymbol{J}$.

Now in § 45 we have already calculated the potential of a given magnetic field $\boldsymbol{H}$ with respect to a magnetic dipole of permanent magnetic moment $\boldsymbol{M}$ situated in it. This potential is the variable part of the expression (230):

$$-(\boldsymbol{M}_x\boldsymbol{H}_x + \boldsymbol{M}_y\boldsymbol{H}_y + \boldsymbol{M}_z\boldsymbol{H}_z) \quad . \quad . \quad . \quad (311)$$

But the magnetic double-layer of permanent moment $\frac{\mu J}{c}$ which, by § 58, is equivalent to the current J, consists of an infinite number of dipoles each of which lies on a surface-element of the layer $d\sigma$ and has the moment:

$$|\boldsymbol{M}| = \frac{\mu J}{c}\, d\sigma \quad . \quad . \quad . \quad . \quad . \quad (312)$$

and has the normal ν to the surface as its axis in the sense given by the direction of the current. Hence by substituting (312) in (311) and summing over all the elements $d\sigma$ of the double-layer we obtain for the required electromagnetic potential:

$$V = -\frac{\mu J}{c}\int H_\nu d\sigma = -\frac{J}{c}\int B_\nu d\sigma \quad . \quad . \quad (313)$$

The potential of a magnetic field with respect to a linear current is thus equal to the negative product of $\frac{J}{c}$ and the flux of induction of the field through a surface bounded by the current or, as we may also say, the number of lines of induction which the conductor carrying the current encircles. We have already seen in equation (53) that this surface integral is independent of the special form of the surface.

Since the tendency of the mechanical interactions between the magnet and the current is also such as to decrease the value of V we can express this briefly by saying that the mechanical forces tend to increase the flux of magnetic induction through the surface bounded by the current. This holds for any displacement of the current and the magnet whatsoever. If the magnet is kept fixed then the current seeks either by rotation or translation, to get into the position in which it encircles in the positive sense as many as possible of the lines of induction that proceed from the magnet. If the wire carrying the current is kept fixed the magnet seeks to move in such a way that it sends as many lines of induction as possible through the surface bounded by the current in the positive sense. For example, this occurs in the case of a circular current when the centre of the magnet is at the centre of the circle and the axis of the magnet coincides with the positive normal of the current.

The fact that the electromagnetic potential V is independent of the special form of the surface σ of the current-circuit may also be expressed in a formula by introducing

the vector-potential $\boldsymbol{A}$ of the magnet by (282) or (298) in place of the induction $\boldsymbol{B}$. By applying Stokes's Theorem (273) we then get:

$$\int \boldsymbol{B}_\nu d\sigma = \oint \boldsymbol{A} \cdot d\boldsymbol{r} \quad . \quad . \quad . \quad . \quad . \quad (313a)$$

and from (313):

$$V = -\frac{J}{c}\oint \boldsymbol{A} \cdot d\boldsymbol{r} \quad . \quad . \quad . \quad . \quad . \quad (314)$$

where the integration is now only to be taken over the curve-elements $d\boldsymbol{r}$ of the conductor. Hence it follows, in view of (292), that for a space-current:

$$V = -\frac{1}{c}\int \boldsymbol{J} \cdot \boldsymbol{A} \cdot d\tau \quad . \quad . \quad . \quad . \quad . \quad (315)$$

§ 70. Having applied the integral law we now come to the differential law of the mechanical electromagnetic effects. Concerning the mechanical force exerted on a magnet pole by a current-element, we obtain this from Biot and Savart's law (204) since the magnetic field-strength produced by the current simultaneously also represents the mechanical force which acts on a positive magnet pole situated in the field.

Again, to find the mechanical force $\boldsymbol{F}$ that is exerted by a given magnetic field $\boldsymbol{H}$ or $\boldsymbol{B}$ on an element $d\boldsymbol{r}$ of a linear conductor in it conveying a current we start out from the total mechanical work:

$$\Sigma \boldsymbol{F} \cdot \delta \boldsymbol{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (316)$$

performed by all the individual current-elements during any infinitely small virtual displacement of the conductor. This work is equal to the decrease of the electromagnetic potential and hence is, by (313):

$$-\delta V = +\frac{J}{c}\delta\left(\int \boldsymbol{B}_\nu d\sigma\right). \quad . \quad . \quad . \quad . \quad (317)$$

Here the variation of the integral may be calculated from a simple geometrical consideration. For since the magnetic field $\boldsymbol{B}$ is invariable the variation of the flux of induction through the whole surface σ reduces to the

flux of induction through the infinitely narrower ring-shaped surface which is formed by the displaced curve and the undisplaced curve and which forms with the original surface σ the varied surface σ. The parallelogram formed by the elements $\delta\boldsymbol{r}$ and $d\boldsymbol{r}$ is an element of this surface, or, written vectorially:

$$[\delta\boldsymbol{r},\, d\boldsymbol{r}]$$

and the flux of induction through it is:

$$\boldsymbol{B}\,.\,[\delta\boldsymbol{r},\, d\boldsymbol{r}].$$

So, integrating over all the elements of the ring surface we get by (317):

$$-\,\delta V = \frac{J}{c}\int \boldsymbol{B}\,.\,[\delta\boldsymbol{r},\, d\boldsymbol{r}] = \frac{J}{c}\int \delta\boldsymbol{r}\,.\,[d\boldsymbol{r},\, \boldsymbol{B}] \quad . \quad (317a)$$

and, equating this to (316), since the individual $\delta\boldsymbol{r}$'s are independent of one another:

$$\boldsymbol{F} = \frac{J}{c}[d\boldsymbol{r},\, \boldsymbol{B}] \quad . \quad . \quad . \quad . \quad . \quad (318)$$

Thus the force is proportional to the sine of the angle formed by the current-element and the line of magnetic induction that passes through it, and this force acts perpendicularly to both. Its direction is indicated in Fig. 9 and can be established by the rule that for a man swimming in the direction of the magnetic line of induction $\boldsymbol{B}$ and looking in the direction of the electric current $\boldsymbol{J}$ the force $\boldsymbol{F}$ acts towards the right. Accordingly the current-element is driven by the force to cut the lines of induction transversely, and this formulation again shows the relationship with the integral law, according to which the current seeks to encircle as many lines of induction as possible.

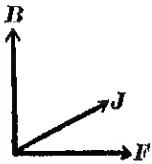

FIG. 9.

§ 71. Concerning the mutual ponderomotive actions of currents on one another, that is, *electrodynamic* actions, it is obvious that the action of a magnetic field on a

current-element is independent of whether the magnetic field is due to a magnet or to a second current. Hence the law of force (318) again applies here.

To set up the electrodynamic law of potential we form the electrodynamic potential W from the electromagnetic potential V by substituting in (315) the expression of the vector-potential (289) of a system of currents in space:

$$W = -\frac{1}{2}\frac{\mu}{c^2}\iint\frac{\boldsymbol{J}\boldsymbol{J}'d\tau d\tau'}{r} \quad . \quad . \quad . \quad . \quad (319)$$

It is necessary to insert the factor $\frac{1}{2}$ because in the double integral the combination of each pair of volume-elements $d\tau$ and $d\tau'$ occurs twice.

This general expression for the electrodynamic potential of a system of conductors in space at the same time contains the effect of a current on its own conductor. For the integration is also to be performed over each two volume-elements $d\tau$ and $d\tau'$ of the same current-filament. The amount for two infinitely near current-elements is not then sufficiently appreciable to come into consideration, on account of the smallness of $d\tau$ and $d\tau'$, in spite of the vanishing of r.

If we express the scalar product $\boldsymbol{J}\,.\,\boldsymbol{J}'$ in terms of the absolute values of the vectors, we may also write:

$$W = -\frac{1}{2}\frac{\mu}{c^2}\cdot\iint\frac{|\boldsymbol{J}|\,.\,|\boldsymbol{J}'|\,.\cos\delta}{r}\cdot d\tau\,.\,d\tau' \quad . \quad (320)$$

where δ denotes the angle between the vectors $\boldsymbol{J}$ and $\boldsymbol{J}'$.

Let us apply the electrodynamic law of potential to the mutual ponderomotive action between two linear currents J and J'. As we wish to neglect the actions of the currents on themselves we need refer $d\tau$ only to the one conductor and $d\tau'$ only to the other. Taking into account (291a) we then obtain from (320), neglecting the factor $\frac{1}{2}$:

$$W = -\frac{\mu}{c^2}JJ'\iint\frac{\cos\delta}{r}dsds' \quad . \quad . \quad . \quad . \quad (321)$$

According to the remark made in connection with (313)

this quantity is equal to $-\frac{J}{c}$ times the flux of magnetic induction which the current J', on account of its magnetic action, sends in the positive direction through a surface bounded by the current, and conversely. In this sense W is proportional to the number of lines of induction which both conductors encircle in common. The ponderomotive forces between the conductors carrying the currents always act in a direction such that the number of lines of induction becomes as large as possible; this may be achieved by translation or rotation. Hence currents that are parallel and in the same direction attract each other whereas parallel and opposite currents repel each other or seek to turn themselves so that they run in the same direction.

§ 72. One theorem is sufficient to characterize completely all that has so far been said about the ponderomotive actions that occur in a system of freely moving magnets and conductors carrying currents. This theorem states that for any virtual infinitely small displacement of the magnets and conductors the mechanical work performed by the ponderomotive forces during this process is equal to the decrease of the total potential:

$$U + V + W = F \quad . \quad . \quad . \quad . \quad (322)$$

where $-\delta U$ represents the virtual work of the magnetic forces, $-\delta V$ that of the electromagnetic forces and $-\delta W$ that of the electrodynamic forces.

The expressions for the potentials V and W are obtained from the relations given above. If, in particular, we are dealing with linear conductors, in which the currents J_1, J_2, . . . are flowing, we have by (313):

$$V = -\frac{1}{c}(J_1 V_1 + J_2 V_2 + \ldots) \quad . \quad . \quad (323)$$

where V_1, V_2, . . . denote the flux of induction of the field produced by the magnets through a surface bounded

by the current 1, 2, . . . Further we have by (320), analogously to (321):

$$W = -\frac{1}{c^2}\left(\frac{1}{2}L_{11}.J_1^2 + L_{12}.J_1J_2 + \frac{1}{2}L_{22}.J_2^2 + L_{13}.J_1J_3 + \ldots\right) \quad (324)$$

where:

$$L_{12} = L_{21} = \mu . \iint \frac{\cos\delta}{r}\, ds_1\, ds_2 \quad . \quad . \quad (325)$$

The coefficients L depend only on the geometrical configuration of the system of conductors, and by (321) $\frac{1}{c}L_{12}J_2$ is equal to the flux of magnetic induction which the current J_2 sends in the positive direction through a surface bounded by the current J_1. The quadratic terms in the sum (324) give the electrodynamic self-potentials of the individual currents, the bilinear terms give the potentials of each pair of different currents on one another. The change in a coefficient L caused by a displacement of the conductor carrying the current gives the work done by the ponderomotive forces between the corresponding currents.

§ 73. The displacement of magnets and conductors which was assumed in calculating the work done by the ponderomotive forces is quite arbitrary. In particular it can also be connected with a deformation of the substances, that is with a bending or distortion of the conductors. According as the corresponding work comes out positive or negative the ponderomotive forces act in the same or in the opposite sense as the deformation in question. But there is an important point to bear in mind. For as we can see from the manner in which we apply the potential law to the interaction between a current and a magnet we must assume the intensity of current to be invariable in every element of every current-filament during the displacement. If, then, the displacement is connected with a deformation of the conductor the current-filaments remain rigidly connected with the substance of the conductor or, as we may also say, the

lines of flow are taken along by the matter in the sense that a material curve which forms a line of flow before the displacement also forms one after the displacement.

It is only when this rule is strictly observed that we can avoid the pit-falls which lead so readily to erroneous applications; such applications have even led some physicists to raise objections to the general validity of the potential law.

An example may serve to illustrate this. Suppose we have an intense homogeneous magnetic field $\boldsymbol{H}$ or $\boldsymbol{B}$ directed vertically upwards, and suppose that in it we have a straight linear conductor OA of length R (Fig. 10) which can be rotated about a fixed point O in a horizontal plane, and which is traversed by a stationary current J. Let us suppose that the current is transmitted as follows: the horizontal circular arc on which the end-point A of the conductor can glide forms the upper rim of the conducting curved surface of a long hollow vertical cylinder which receives the current at A and conducts it to a great distance below where the electromotive force is supposed to be applied, which again drives the current through the conducting axis of the cylinder right up to the turning-point O. The question is whether the radius R, which can be rotated and which is traversed by the current, experiences a turning-moment from the magnetic field and, if so, in which direction and of what amount?

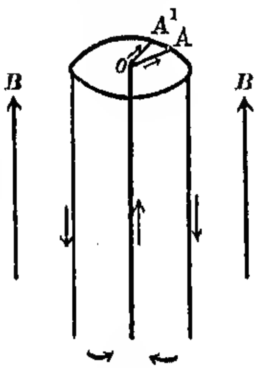

Fig. 10.

We first solve the problem by means of the potential law and then by the law of force. As we are here concerned with an electromagnetic action the potential V is the only one of the three to come into account. The following fallacious line of reasoning immediately suggests itself. The general expression for V is given by (315) and

the value of this expression is certainly independent of the position of the radius in its plane. For if we were to introduce a co-ordinate system whose x-axis is the radius R and whose z-axis is the vertical through O then on account of the conditions of symmetry with respect to this system the position of all the lines of flow is quite definite, independently of whether the radius R ends in A or in A'. Consequently—so we should be tempted to conclude—for an infinitely small rotation of the radius we have $\delta V = 0$ and no mechanical action at all takes place.

This conclusion is false, because in the above calculation of δV the condition has been transgressed that in this virtual rotation the current-filaments must remain rigidly connected with the substance of the conductor. For since the conducting cylinder is at rest, the lines of flow that are in it also remain completely unchanged during the variation, and the course of flow must be assumed to be as follows in the displaced position of the radius R: from O to A', then linearly to A, and from A exactly as before downwards through the curved surface of the cylinder and upwards along the axis of the cylinder. In the present case the virtual change of the flow is quite different from the actual change that occurs when the radius is rotated.

If by (315) we first form the expression for V for the flow before the displacement and then for the flow after the displacement, and subtract the first value from the second, then, since the vector-potential $\mathbf{A}$ which arises from the magnetic field is invariable, all the terms of the two integrals that refer to the parts of the conductors that are at rest cancel out, and all we have to do is to integrate the expression:

$$\delta V = -\frac{J}{c}\oint \mathbf{A} \, . \, d\mathbf{r}$$

linearly from O over A' and A back to O. But on account of (313a) this line-integral is equal to the flux of induction of the magnetic field through the narrow triangle $OA'A$ in the sense of the normal which corresponds to the direction of integration, in the case of Fig. 10 downwards;

this is opposite to the direction of the magnetic field and so is negative. If ζ is the infinitely small angle of rotation, the surface of the triangle $OA'A$ is $\frac{R^2}{2}\cdot\zeta$, and accordingly we have:

$$\delta V = \frac{J}{c}\frac{R^2}{2}\zeta\cdot \boldsymbol{B}. \quad . \quad . \quad . \quad . \quad . \quad (326)$$

On the other hand if we denote the mechanical turning moment by N the mechanical virtual work is $N\,.\,\zeta$ and consequently, since the work is equal to the decrease of potential:

$$N = -\frac{J}{c}\frac{R^2}{2}\cdot \boldsymbol{B}. \quad . \quad . \quad . \quad . \quad . \quad (327)$$

Thus the turning moment acts in the direction from A' to A.

We arrive at the same result by applying the law of force (318) to the present case. The turning moment of the given magnetic field on the movable radius R is built up by summing up the turning moments that act on the individual elements dr of the radius. Now by (318) the force on an element dr is $\frac{J}{c}\cdot\boldsymbol{B}\,.\,dr$, and so the turning-moment is the product of this expression with r, and the resultant turning-moment is:

$$N = -\frac{J}{c}\boldsymbol{B}\int_0^R r dr = -\frac{J}{c}\boldsymbol{B}\frac{R^2}{2}$$

as above. The sense of the rotation is towards the right for an observer who is swimming with the lines of induction, that is, who is standing upright at O and is looking in the direction of the radius R traversed by the current; hence the negative sign of N.

The same danger of drawing fallacious conclusions and the same need for caution clearly occurs whenever we are dealing with conductors that can glide over one another.

Reverting to the case considered in § 72 of a system of arbitrary movable magnets and linear currents $J_1, J_2, \ldots$

we now form the expression for the ponderomotive work A performed during any virtual infinitely small displacement of the system. If we pay due regard to the invariance of the current-strengths we obtain for this work the expression:

$$
\begin{aligned}
A &= -\delta U - \delta V - \delta W \\
&= -\delta U + \frac{1}{c}(J_1\delta V_1 + J_2\delta V_2 + \ldots) \\
&\quad + \frac{1}{c^2}\left(\frac{1}{2}J_1^2\delta L_{11} + J_1J_2\delta L_{12} + \frac{1}{2}J_2^2\delta L_{22} + J_1J_3\delta L_{13} + \ldots\right) \quad (328)
\end{aligned}
$$

From this all the mechanical actions within the system can be derived.

PART THREE

QUASI-STATIONARY AND DYNAMICAL PROCESSES

CHAPTER I

QUASI-STATIONARY PROCESSES WITH CLOSED CIRCUITS

§ 74. IN this last part of the present book where we turn our attention to the general case of electrodynamics, namely, non-stationary processes, our first impression is one of surprise at the enormous complexity of the problems here to be solved. Thus the case of the motion of an invariable magnet in an evacuated space, although it seems very simple at first sight, really represents a very intricate process. For, since on account of the motion of the magnet the magnetic intensity of field $\boldsymbol{H}$ produced by it at a definite point of the vacuum is variable with respect to time, by (31b) an electric field $\boldsymbol{E}$ also exists at this point, which will likewise be variable with respect to the time and so will induce variable electric charges in the magnet, and so forth.

One way of escape out of this maze of mutual actions—a way which gives a comprehensive survey of the decisive interactions among them—is obtained if we follow the course, already once previously adopted in the case of electrostatics in § 40, of first restricting ourselves to investigating processes that occur so slowly that at every moment the whole electromagnetic field may be regarded as stationary; that is, we consider only quasi-stationary processes. The great advantage obtained by this simplification is that the number of variables which determine the state of the system of bodies in question and whose changes in time therefore uniquely determine the course of the process is essentially limited. For example, in the case of a slowly variable quasi-stationary current in a linear con-

ductor at rest the whole state depends on a single variable, namely the current-strength, in that the magnetic field in the surrounding insulator everywhere has exactly the constitution which it should have if the momentary value of the current-strength were constant, and the process is to be regarded as known in all its parts if the current-strength has been determined as a function of the time.

Before commencing our treatment of quasi-stationary processes we shall pause for a moment to consider whether in the case of a definitely given system of bodies there is not a criterion which can be formulated quantitatively to decide whether a process may be regarded as quasi-stationary. If we are dealing with moving bodies a criterion of this kind suggests itself at once: it is the condition that the velocities involved should be small compared with the critical velocity C. But this condition is not sufficient, for in conductors at rest certain electrodynamic processes can also occur which may not be classified as quasi-stationary. In § 89 we shall meet with an easily visualized relation between the magnitudes for these cases which can be applied as a general criterion for the quasi-stationary character of a process.

Among quasi-stationary processes there are some which are distinguished by being particularly simple. We shall discuss them first. In these processes all the electric currents are closed or, expressed more accurately, the flow of electricity is everywhere tangential to the surfaces of the conductors. The simplification effected by this condition is two-fold. Firstly, in the case of closed currents the electric charges that present themselves at the surfaces of the conductors and hence also the corresponding electric intensities of field are everywhere so small that the resulting electric energy of the field can be neglected entirely in comparison with the magnetic energy caused by the currents. Secondly, we can immediately apply the formulae set up in the preceding chapters for the magnetic effects of closed stationary currents to calculate the magnetic energy which now alone remains in

the present case. In this way, by applying the energy-principle, we arrive at definite laws for the processes in question.

§ 75. Let us first calculate the magnetic energy E of a quasi-stationary field which is composed of any arbitrary number of movable permanent magnets and conductors of currents embedded in a common insulator which extends to infinity in all directions. The magnetic intensity of field $\boldsymbol{H}$ at any point of any substance is composed of two components: of the field-strength $\boldsymbol{H}_m$ which arises from the magnets alone and depends only on their momentary position, and of the field-strength $\boldsymbol{H}_i$ which arises from the currents alone and depends only on the momentary positions of the conductors and the momentary flow of electricity in them. Thus:

$$\boldsymbol{H} = \boldsymbol{H}_m + \boldsymbol{H}_i \quad . \quad . \quad . \quad . \quad . \quad (329)$$

The vector $\boldsymbol{H}_m$ is derived from the potential function ϕ of the permanent magnets:

$$\boldsymbol{H}_m = -\operatorname{grad} \phi \quad . \quad . \quad . \quad . \quad . \quad (330)$$

whereas the vector $\boldsymbol{H}_i$, which has no scalar potential, can, by (284), be referred back to the vector-potential $\boldsymbol{A}$ of the currents:

$$\boldsymbol{H}_i = \frac{1}{\mu} \operatorname{curl} \boldsymbol{A} \quad . \quad . \quad . \quad . \quad . \quad (331)$$

Since we assume μ to have the same value in all substances (§ 55) $\boldsymbol{H}_i$ is continuous throughout. In view of the remark at the end of § 35 concerning the magnetic energy when permanent magnets are present we have for the magnetic energy of the whole system:

$$E = \frac{\mu}{8\pi}\int \boldsymbol{H}^2 d\tau$$
$$= \frac{\mu}{8\pi}\int \boldsymbol{H}_m^2 d\tau + \frac{\mu}{4\pi}\int \boldsymbol{H}_m \cdot \boldsymbol{H}_i d\tau + \frac{\mu}{8\pi}\int \boldsymbol{H}_i^2 d\tau \quad . \quad . \quad (332)$$

Let us consider the three parts of this expression separately. The first part represents the magnetic

energy or the magnetic self-potential of the permanent magnets: we designate it as previously by U. By using (330) we can transform the second part; namely, since ϕ and $\boldsymbol{H}_i$ are continuous:

$$\int \boldsymbol{H}_m \,.\, \boldsymbol{H}_i d\tau = \int \phi \,.\, \operatorname{div} \boldsymbol{H}_i d\tau.$$

On account of (331) this expression vanishes. Finally the third part when correspondingly transformed by (331) gives us:

$$\frac{\mu}{8\pi}\int \boldsymbol{H}_i^2 d\tau = \frac{1}{8\pi}\int \operatorname{curl} \boldsymbol{A} \,.\, \boldsymbol{H}_i d\tau$$

$$= \frac{1}{8\pi}\int \boldsymbol{A} \,.\, \operatorname{curl} \boldsymbol{H}_i d\tau \quad . \quad . \quad (332a)$$

and, by (241) and (289), we immediately have:

$$\frac{1}{2c}\int \boldsymbol{A} \,.\, \boldsymbol{J} d\tau = \frac{\mu}{2c^2}\iint \frac{\boldsymbol{J} \,.\, \boldsymbol{J}'}{r}\, d\tau\, d\tau'$$

in which double integral every volume-element of the whole space belongs both to the $d\tau$ and to the $d\tau'$.

Comparison with (319) shows that this expression is nothing other than the negative electrodynamic potential W of the current system in space. We obtain as the total value of the magnetic energy of all the magnets and all the currents:

$$E = U - W \quad . \quad . \quad . \quad . \quad . \quad . \quad (333)$$

At first sight this result, according to which the total magnetic energy is simply composed additively of the energy U of the magnets and the negative potential W of the currents, and involves no term due to the interaction of magnets and currents, may appear rather surprising and striking. In particular the question forces itself on us, whence does it come that the value of E differs from the value, expressed in (322), of the total potential F of the magnets and the currents, which we obtained by replacing every current by its equivalent magnetic shell or double-layer? The physical ground of this contrast between E and F consists in the fact that although a

current can be fully replaced by a magnet so far as the effects which emanate from it are concerned yet, in spite of this, it cannot be completely identified with a magnet. For the manner in which a current reacts to a magnetic field in which it is situated corresponds with the behaviour of the equivalent magnet only with regard to the ponderomotive actions, in accordance with the principle of action and reaction, but not, as we shall see, with respect to its electromotive effects. It is on this difference between a current and its equivalent magnet that the difference between E and F is founded.

To distinguish the total electromagnetic energy E from the potential F which is all-important only for the mechanical work, E is often called the "free energy." We cannot here discuss the general relationship between the functions E and F. But we may, by way of an example, state that for a field of permanent magnets and linear currents J_1, J_2, J_3 . . . the following relation holds:

$$E = F - \left(J_1\frac{\partial F}{\partial J_1} + J_2\frac{\partial F}{\partial J_2} + \ . \ . \ .\right) \quad . \quad . \ (333a)$$

which can be verified immediately by means of the equations (322), (323), (324). It will be observed that it is of precisely the same form as the equation I (407) between the energy E and the Lagrange function L.

§ 76. We shall now make use of the expression we have found to solve our problem. For this purpose we shall apply the principle of conservation of energy to a series of typical cases, beginning with the simplest, that of a single linear conductor.

To fix our ideas we shall begin with a galvanic chain which is open initially and which has the given electromotive force E [1] and the given resistance w. We require to know what is the value of the current-strength J expressed as a function of the time t, starting from the moment $t = 0$, when the chain is closed, so that $J = 0$ for $t = 0$. To obtain a clearer insight into the way in

[1] This must not be confused with the energy E.

which the quantities characteristic for the process depend on the time t we shall not assume that the conductor is at rest but that it is flexible and extensible by supposing, say, that a spiral is included in the circuit, whose turns can approach or recede from each other. Through this we achieve that in the expression (324) for the magnetic energy of the current:

$$-W = \frac{1}{2c^2} LJ^2 \quad . \quad . \quad . \quad . \quad (334)$$

the coefficient L which is dependent on the mutual position of the parts of the conductor can also depend on the time.

The principle of the conservation of energy demands that the sum of all the energies that occur in the system in question should have an invariable value, or its change in the element of time dt should vanish. In the present case we have to take into account, besides the magnetic energy expressed in (334), also the mechanical, the thermal and the chemical energy.

By the laws of mechanics I (393) the change that occurs in the mechanical energy (kinetic and elastic energy of the moving and the deformed parts of the conductor) in the time dt is equal to the ponderomotive work A performed by the current, and hence, by (328), in our present case:

$$A = \frac{1}{2c^2} J^2 dL \quad . \quad . \quad . \quad . \quad . \quad (334a)$$

when dL denotes the change in L during the time dt. Also, by (304) and (308), the thermal and the chemical energy experience the change:

$$dt \, . \, J^2 w - dt \, . \, J \, . \, E \quad . \quad . \quad . \quad . \quad (335)$$

in the time dt.

From the last three expressions we therefore get by the energy principle:

$$d\left(\frac{1}{2c^2} LJ^2\right) + \frac{1}{2c^2} J^2 dL + dt(J^2 w - JE) = 0$$

or:

$$\frac{1}{c^2}\frac{d(LJ)}{dt} + wJ - E = 0 \quad . \quad . \quad . \quad (336)$$

a relation which can be used to calculate J as a function of the time t. To visualize its physical meaning we often write it in the form:

$$J = \frac{E - \frac{1}{c^2}\frac{d(LJ)}{dt}}{w} \quad . \quad . \quad . \quad . \quad (337)$$

in which it runs closely parallel to the form of Ohm's Law for a stationary current. Comparison shows that Ohm's law can also be applied in the present case provided that we assume that besides the electromotive force E furnished by the contact potentials a second electromotive force also acts, whose value is:

$$E' = -\frac{1}{c^2}\frac{d(LJ)}{dt} \quad . \quad . \quad . \quad . \quad (338)$$

and which is called the induced electromotive force, or here, the electromotive force of self-induction. As we see, this consists of two parts: the one is due to the variability of the coefficient of self-induction or the "inductivity" L, that is, on the mutual displacement of the parts of the conductor; the other is due to the variability of the current-strength J.

But we must not regard this extension of Ohm's Law to non-stationary currents as any more than a purely formal way of re-writing the relationship (336). In fact, doubts assail us as to whether we are justified in calling the quantity (338) an electromotive force. For since a force can be regarded as a cause, we can arrive at the idea that the expression with the differential coefficient $\frac{dJ}{dt}$ is the cause of the value of J; this is nonsense, because the cause must precede the effect in time. $\frac{dJ}{dt}$ is not the cause of J; rather, the reverse is the case. The value of J, or, more precisely, its deviation from the stationary

value $\frac{E}{w}$ is the cause of $\frac{dJ}{dt}$. The analogy with mechanical processes is therefore brought out more strikingly if we write the quantities in (336) with the reverse sign and then regard the first term as inertial resistance (I, § 66), the second as frictional force, and the third as a driving force.

If in particular we regard the conductor as at rest, that is, L a constant, the differential equation (336) can be integrated and we get, taking into consideration that $J = 0$ for $t = 0$:

$$J = \frac{E}{w}\left(1 - e^{-\frac{c^2wt}{L}}\right) \quad . \quad . \quad . \quad . \quad (339)$$

From this we see how the current-strength proceeds from its initial value 0 to the stationary value $\frac{E}{w}$. This is as a rule a very rapid process on account of the great numerical value of c. We also see how this process is accelerated by increasing the resistance w, and retarded by increasing the self-induction L. This can be formulated by saying that when the circuit is closed (at "make") the stationary current $\frac{E}{w}$ does not establish itself instantaneously but rather that an extra current, here an induced current at "make," becomes superposed in a direction contrary to that of the stationary current and has the full value of the latter at the beginning but then rapidly decreases to zero. Corresponding to this induced current at "make," there is an equally great but oppositely directed current—that is one in the same direction as the main current—which is called the induced current at "break." This current presents itself when, with the stationary current originally flowing in the circuit, the electromotive force E is suddenly eliminated at the moment $t = 0$ without the circuit being broken. In this case the total energy is originally electromagnetic and in the course of the process becomes completely transformed into Joule heat.

§ 77. We next consider the interactions between a moving linear conductor of resistance w and a moving permanent magnet. We shall neglect the self-induction of the conductor here and consequently take L as correspondingly small. By (333) the magnetic energy then reduces to a constant, since the energy U of the magnet does not change with the time, and the change of the mechanical energy (kinetic energy of the conductor and the magnet) which occurs in the time dt is, by (328):

$$A = \frac{1}{c} J dV_1 \quad . \quad . \quad . \quad . \quad . \quad (340)$$

whereas the corresponding change of the thermal and the chemical energy is again represented by (335). By the energy principle the sum of all the changes of energy is equal to zero, hence:

$$\frac{1}{c} J dV_1 + dt(J^2 w - JE) = 0$$

or:

$$\frac{1}{c}\frac{dV_1}{dt} + Jw - E = 0 \quad . \quad . \quad . \quad (340a)$$

If we again write this relation in the form of Ohm's Law:

$$J = \frac{E + E'}{w} \quad . \quad . \quad . \quad . \quad . \quad (341)$$

and denote the induced electromotive force by E', then:

$$E' = -\frac{1}{c}\frac{dV_1}{dt} \quad . \quad . \quad . \quad . \quad (342)$$

is the electromotive force induced in the conductor by the relative motions of the conductor and the magnet. As we see, here, too, the induced electromotive force depends neither on the contact potentials E that occur in the circuit nor on the current-strength J in it, but only on the relative motion of these two bodies. For by (323) the value of V_1 represents the magnetic flux of induction of the field produced by the magnet through a

surface bounded by the conducting circuit, or the number of lines of induction originating in the magnet, which the conductor encircles, being positive when their direction corresponds to the sense in which the current-strength is reckoned as positive. This quantity does not change when the conductor and the current remain rigidly connected during their motion.

According to (342) the sense of the induced electromotive force is always contrary to the increase of the magnetic flux of induction just mentioned, or, as we may also say, the induced electric current $\frac{E'}{w}$ always flows in the sense that makes the ponderomotive work (340) negative, so that the kinetic energy of the motion decreases. Hence the mechanical forces caused by the induced current always tend to oppose the motion (Lenz's Rule), in the manner of frictional forces. Here again we see the similarity between the nature of Joule heat and frictional heat.

§ 78. After the last special examples we shall now immediately proceed to the more general case of the motion of a system of any arbitrary number of linear currents J_1, J_2, J_3, . . . and any number of permanent magnets. In this case, too, the total energy is made up of the magnetic, the mechanical, the thermal and the chemical energy. By (333) the change that occurs in the magnetic energy in the time dt is here:

$$dU - dW \quad . \quad . \quad . \quad . \quad . \quad . \quad (343)$$

in the expression for W given by (324) the coefficients L and also the current-strengths J are here to be regarded as dependent on the time t.

The change in the mechanical energy of the system is expressed by the value for the work A which results from (328) when we write the time-differential d everywhere in it in place of the variation δ. Lastly, by introducing the induced electromotive force E' from (341), we may write for the thermal and chemical change of

energy, which is obtained by summing up (335) over all the conductors :

$$dt \,.\, (J_1E'_1 + J_2E'_2 + \ldots) \quad . \quad . \quad . \quad (344)$$

and by applying the energy principle to the whole system in question, the expression :

$$dU - dW + A + dt \,.\, (J_1E'_1 + J_2E'_2 + \ldots) = 0 \quad (345)$$

This equation is, in fact, satisfied identically in all its parts if we substitute for the electromotive forces E'_1, E'_2, . . . induced in the conductors the expressions which necessarily follow from the values found above for the special cases. For the relation (342) must also hold here if in place of V_1 we write the total magnetic flux of induction through a surface bounded by the conductor 1, since it is a matter of indifference whether the magnetic field in which the conductor is situated is due to magnets or to currents. Now the total flux of induction sent by the magnets and the other currents through a surface bounded by the conductor 1 is, according to the remarks made in connection with (323) and (325) :

$$V_1 + \frac{1}{c}(L_{12}J_2 + L_{13}J_3 + \ldots).$$

If we substitute this in (342) instead of V_1 and also take into account the self-induction (338) we obtain for the total electromotive force induced in the conductor 1 :

$$E'_1 = -\frac{1}{c}\frac{d}{dt}\left(V_1 + \frac{L_{11}J_1 + L_{12}J_2 + L_{13}J_3 + \ldots}{c}\right) \quad (346)$$

and corresponding expressions for the quantities E'_2, E'_3, . . .

From this we see that electromotive forces can be induced in a circuit either by motions of magnets and currents, which change the quantities V and L, or by changes in the current-strengths J, but *only in so far as the total magnetic flux of induction through a surface bounded by the circuit becomes changed.*

If we substitute the expressions (346) in the condition

(345), and if we take into consideration the values of dW and A, we find by direct calculation that the relation is satisfied identically. This confirms the inner relationships contained in the theory that has been developed.

§ 79. The law of induced electromotive forces stated in equation (346) can be immediately extended, in view of its physical meaning as above described, to the case where the inducing magnetic field $\boldsymbol{B}$ is caused not solely by magnets and linear currents but also by space currents. Accordingly the most general and at the same time the simplest form of the expression for the electromotive force induced in any closed linear circuit is:

$$E' = -\frac{1}{c}\frac{d}{dt}\int \boldsymbol{B}_\nu d\sigma \quad . \quad . \quad . \quad . \quad (347)$$

This contains all the cases hitherto discussed as special cases.

In applying (347) it is obvious but nevertheless remarkable that the closed curve which bounds the surface σ is formed at the time $t + dt$ by the same material points as at the time t. Hence it follows, for example, for the case considered in § 73 of a radius OA (Fig. 10) that can be rotated in a magnetic field OA, that when the radius is rotated from A and A' through the infinitely small angle ζ the change of the magnetic flux of induction which must be inserted in (347) is equal to the flux of induction through the infinitely narrow triangle OAA', and that hence the electromotive force:

$$\frac{1}{c}\cdot \boldsymbol{B}\cdot\frac{R^2}{2}\frac{\zeta}{dt}$$

will be induced in the rotating radius in the direction from O to A, which agrees with Lenz's rule, since a current flowing from O to A causes rotation in the reverse direction, namely from A' to A.

In the form (347) the law of induced electromotive forces presents itself as an integral law. The question naturally arises as to the form of the corresponding

differential law, as this will inform us about the electric intensities of field induced in the individual infinitely small parts of the conductor. If we denote these intensities by $\boldsymbol{E}$, then we have for the electromotive force in the whole closed circuit:

$$E' = \oint \boldsymbol{E} \cdot d\boldsymbol{r} \quad . \quad . \quad . \quad . \quad . \quad (348)$$

where the integration of the scalar product of the intensity of field $\boldsymbol{E}$ and the element $d\boldsymbol{r}$ of the conductor is to be performed over the closed curve of the circuit.

To be able to derive the value of $\boldsymbol{E}$ from (347) we must also transform the right-hand side of this equation into a line-integral in terms of $d\boldsymbol{r}$. Now the time change of the magnetic flux of induction through the surface σ bounded by the circuit is conditioned by two different circumstances: firstly, by a change in the magnetic field, secondly, by a movement of the boundary of the circuit. Hence the whole change of the integral (347) in the element of time dt is composed additively of one part:

$$dt \cdot \int \dot{\boldsymbol{B}}_\nu d\sigma \quad . \quad . \quad . \quad . \quad . \quad (348a)$$

or, by (313a)

$$dt \cdot \int \dot{\boldsymbol{A}} \cdot d\boldsymbol{r} \quad . \quad . \quad . \quad . \quad . \quad (349)$$

which denotes the time change of the magnetic flux of induction when the boundary curve is held fixed, and another part which denotes the time change of the magnetic flux of induction in the constant magnetic field $\boldsymbol{B}$ in consequence of the displacement of the boundary and which, if the expressions (317) and (317a) are equated, assumes the following value:

$$\int \delta\boldsymbol{r} \cdot [d\boldsymbol{r}, \boldsymbol{B}] \quad . \quad . \quad . \quad . \quad . \quad (349a)$$

where $\delta\boldsymbol{r} = \boldsymbol{q} \cdot dt$ denotes the displacement of the point $\boldsymbol{r}$ of the conductor moving with the velocity $\boldsymbol{q}$ in the

time dt. Substituting this value we may also write the last expression in the form :

$$dt \int q \, . \, [dr, B] = dt \int dr \, . \, [B, q] \quad . \quad . \quad (350)$$

so that by adding together (349) and (350) and substituting in (347) we get for the electromotive force induced when the circuit is closed :

$$E' = \oint E \, . \, dr = -\frac{1}{c}\int \dot{A} dr - \frac{1}{c}\int dr \, . \, [B \, . \, q]$$

and for the local intensity of field in an element of the conductor :

$$E = -\frac{1}{c}\dot{A} - \frac{1}{c}[B, q] - \operatorname{grad}\phi \quad . \quad . \quad (351)$$

where ϕ denotes a certain scalar uniform and continuous function of the reference-point. So long as we are concerned only with the electromotive force E' in the whole closed circuit the value of the function ϕ is quite immaterial. If, however, we enquire into the value of the actual intensity E of the electric field at a definite point of the conductor, then, in so far as the other quantities $\dot{A}$, B and q are known, ϕ also has a perfectly definite value which can be calculated from the condition div $E = 0$ and for the special boundary conditions imposed in the case in question.

If the local intensity of field E is given by (351), this equation does not represent a pure differential law, since the potentials A and ϕ occur on the right-hand side. We can, however, eliminate these quantities simply by forming the curl on both sides of (351). In view of (282) the following relation then results :

$$\operatorname{curl} E = -\frac{1}{c}\dot{B} - \frac{1}{c}\operatorname{curl}[B, q] \quad . \quad . \quad (351a)$$

in which only field-quantities now occur.

The differential law (351) or (351a), respectively, refers the field-strength produced in an element of a conductor

to two different causes, namely, to the local time variation of the field in which the element of the conductor is situated and to the motion of this element. The change in the field acts in accordance with the local change of the vector-potential A, the motion of the element of the conductor works in such a way that the electric intensity of field generated is proportional to the area of the parallelogram formed by the velocity q and the local magnetic induction B and is perpendicular to this parallelogram in the sense indicated in Fig. 11, namely so that for an observer swimming in the direction of the magnetic induction B and looking in the direction of the velocity q the electric intensity of field E acts towards the right. If this intensity of field E generates a current J in the same sense in the conductor, the magnetic field B acts on the current with a ponderomotive force F, which, by (318) (Fig. 9) is opposite to the velocity q of the conductor, and so seeks to hinder it, which is in conformity with Lenz's rule (§ 77).

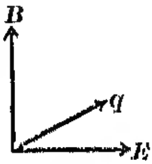

FIG. 11.

The two causes of electromotive induction (or induced electromotive forces), the change in the field and the motion of the conductor, are thus not so different by nature as might be assumed at first sight. For the one cause can to a certain extent replace the other. For example, if we recollect that by (347) the electromotive force induced in a closed linear conductor by the relative motion of a permanent magnet in its vicinity depends only on this relative motion, it is clear that physically it makes no difference whether the conductor is at rest and the magnet is moved or whether the magnet is at rest and the conductor is correspondingly moved. In the former case, however, the cause of the induced field-strength E in an element of the conductor is expressed in the term involving $\dot{A}$, in the latter case only in the term involving q. The fact that in both cases the same value comes out for E in each element of the conductor

is due to the addition of the function ϕ in (351), as closer examination shows. We cannot, however, go into this further here (but cf. § 99 below).

§ 80. Now that we know the differential law (351) for the induced electromotive force we are able to calculate the strength of the induced electric field at every point also for *spatially* extended bodies, and, moreover, since the equation contains no material constants, for insulators as well as for conductors. We can obtain, for example, the electrical excitation and the course of the current in a copper disc which is in motion between the poles of a magnet, or in a copper enclosure containing a vibrating magnet needle. These so-called Foucault currents obey Lenz's rule in that they always flow in such a sense that in virtue of the mechanical forces that act between them and the magnet the motion is obstructed, just as is the case with friction.

As another example we consider the process of excitation of a current in a closed circuit of finite cross-section, say by inserting a galvanic battery. By § 76 the current-strength grows more or less rapidly in virtue of self-induction from the value zero to its stationary value. But the current-strength is not the same simultaneously at all points of a cross-section, rather the self-induction acts on the internal points much more strongly than on the external points situated near the insulator. For, as a glance at the structure of the expression (289) shows, the vector-potential A, through whose change in time the self-induction is caused, by (351), has greater values in the interior of the current-stream than at its edge. For this reason the increase of the current occurs much more slowly in the interior of the conductor than at the surface, and it can happen in the case of rapidly alternating currents in a cylindrical conductor that the inner parts of the conductor remain almost free of current (*stromlos*), that is, that there is no appreciable difference when a hollow cylinder is substituted for the massive cylinder (*Skin effect*).

§ 81. We can also make several other deductions of fundamental importance from the equation (351). For since there is only one kind of electric intensity of field the electric intensity of field induced in an insulator causes, like a statical intensity of field, an electric polarization corresponding to its dielectric constant (§ 26); that is *an insulator that moves in a magnetic field becomes dielectrically polarized to the extent given by* (146).

If, further, the body that moves in the magnetic field carries an electric charge e, as, for example, a small charged sphere, the induced electric intensity of field $\boldsymbol{E}$ by (214) exerts a ponderomotive force on the body of amount:

$$\boldsymbol{F} = e \cdot \boldsymbol{E} = -\frac{e}{c}[\boldsymbol{B}, \boldsymbol{q}] = \frac{e}{c}[\boldsymbol{q}, \boldsymbol{B}] \quad . \quad . \quad (352)$$

If we compare this expression with that for the mechanical force (318) which the same magnetic field exerts on a current-element $J \cdot d\boldsymbol{r}$ situated at the same point we get full agreement by setting:

$$J d\boldsymbol{r} = e\boldsymbol{q} \quad . \quad . \quad . \quad . \quad . \quad . \quad (353)$$

That is, the force exerted by a magnetic field on a moving point-charge is exactly equal to that exerted on a current-element flowing in the direction of the motion and is of corresponding intensity and length (this is exemplified in the deflection of cathode rays, α- and β-rays). Hence we also call a moving electric charge a "convection current" and can enunciate the theorem that a convection current is equivalent to a conduction current. On account of the mechanical principle of action and reaction this equivalence also applies to the mechanical action which a convection current exerts on a magnet (Rowland effect) and consequently also to all the other magnetic effects of the current. Hence we can say quite generally that *a convection current has the same magnetic properties as the conduction current determined by* (353).

All the theorems here deduced which refer to the motion of a body in a constant magnetic field are equally valid according to the Principle of Relativity, as already emphasized in § 79, for a body at rest towards which a magnet moves, that is, for a stationary body in a variable magnetic field.

CHAPTER II

QUASI-STATIONARY PROCESSES IN THE CASE OF UNCLOSED CURRENTS

§ 82. The transition from closed to unclosed currents, of which we have an example in the current discharge of an electric condenser, is accompanied by the two-fold change, as remarked at the end of § 74, that first the electric energy of field may no longer be neglected in comparison with the magnetic energy of field, and secondly that the expression for the magnetic energy can no longer be directly taken over from that which holds for stationary currents.

Let us first deal with the *electric* energy. At every point of the surface of a conductor, where the current-flow has a normal component that differs from zero and hence comes partly to an end in the transition to the insulator, a more or less considerable density of electric charge h accumulates, in accordance with the boundary condition (45):

$$\frac{\partial h}{\partial t} = -J_\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad (354)$$

since J'_ν vanishes for the insulator. If we multiply this equation by the surface-element $d\sigma$ and integrate over any arbitrary portion of the surface, the boundary condition assumes the form:

$$\frac{\partial e}{\partial t} = \dot{e} = J \quad . \quad . \quad . \quad . \quad . \quad . \quad (355)$$

where e now denotes the whole electric charge situated on the portion of the surface and J denotes the total current-strength which is directed towards this portion

O

of the surface and ends there, whereas in (354) J_ν denoted the inwardly directed component of the current-flow.

The charge accumulated at the surface will produce an electric field that varies with the time. But the assumption that the process is quasi-stationary will allow us to regard this field as statical at any instant and to calculate its potential function just as if the momentary charges at all points of the surfaces of the conductor were permanently present there. This assumption corresponds completely with that made in the preceding chapter with reference to the magnetic energy, namely that the magnetic field in its whole extent instantaneously copies the fluctuations of the current-strengths.

Thus if we imagine an arbitrary system of conductors in an insulator of dielectric-constant ϵ stretching to infinity in all directions, the electric potential function will at every moment have the value:

$$\phi = \frac{1}{\epsilon}\int\frac{h d\sigma}{r} \quad . \quad . \quad . \quad . \quad . \quad (356)$$

in the whole of space, the integration being performed over the surfaces of all the conductors. From this the simultaneous electric intensity of field and electric energy of field immediately follow, as in an electrostatic field.

§ 83. It is a little more difficult to express the *magnetic* energy of unclosed currents in terms of the current-strengths. For the calculations made for closed currents in § 62 here leave us in the lurch, as the following simple reflection shows. If we wished also to introduce here as a first approximation the relationship (241) there used between the current-density $\boldsymbol{J}$ and the magnetic intensity of field $\boldsymbol{H}$ we should, by applying Stokes's Law, also obtain the relationship (274) again, which connect the magnetic work along a closed curve with the flow of electricity through any surface bounded by the curve. But this relationship has no meaning in the case of unclosed currents. For the flow of electricity just mentioned is not then independent of the form of the surface.

For example, it is equal to the total current-strength if the closed curve lies entirely within the insulator and if the surface cuts the conductor traversed by the current; but it is equal to zero if we place the surface around the end of the current, keeping the boundary curve fixed, and so make it lie entirely in the insulator. The reason for this break-down of equation (241) consists in the fact that in the case of unclosed currents although the volume divergence (50) vanishes the surface divergence (45) of the current-density $\boldsymbol{J}$ does *not* vanish, as we have seen in (354).

To avoid the difficulty just quoted we must replace the equation (241), which as we see can be used only for closed currents, by a more general relationship. We can obtain such a relationship by referring back to the general field equations (31*a*), which may be written in the form:

$$\operatorname{curl} \boldsymbol{H} = \frac{4\pi}{c}\left(\boldsymbol{J} + \frac{\dot{\boldsymbol{D}}}{4\pi}\right) \quad . \quad . \quad . \quad (357)$$

This expression, which is more general than that given by the simpler relationship (241), may be formulated as follows: in the case of unclosed currents the magnetic field is to be calculated by the same method as for closed currents, only that the current-density $\boldsymbol{J}$, which corresponds to the *conduction current*, must be supplemented by another current-density $\frac{\dot{\boldsymbol{D}}}{4\pi}$, which is called the density of the *displacement current*. This is determined by the time-change of the electric field, in particular, of the electrical induction.

§ 84. By introducing the displacement current we can carry out exactly the same calculation as was done in § 62 for closed currents by now writing in place of the current-density $\boldsymbol{J}$ of the conduction-current the current-density of the whole current:

$$\boldsymbol{J} + \frac{\dot{\boldsymbol{D}}}{4\pi} = \boldsymbol{E} \quad . \quad . \quad . \quad . \quad . \quad (358)$$

It is clear that the contradiction above objected to now vanishes, since for the vector E both the volume divergence, by (36), and also the surface divergence, by (34), are equal to zero, and in this sense we may state the theorem that Maxwell's electrodynamics deals only with closed currents.

Thus we obtain for the vector-potential of the total flow, by (289):

$$A^* = A + \frac{\mu}{4\pi c} \cdot \int \frac{\dot{D}'}{r} d\tau' \quad . \quad . \quad . \quad . \quad (359)$$

where, as hitherto, we use the symbol A to refer to the conduction current alone, so that the equation (289) remains valid for A whereas on the other hand instead of (283) we now have div $A^* = 0$.

The additional term in (359) can be put in a more convenient form. The x-component of this vector runs, if we introduce the electric potential function ϕ in place of the electric induction D:

$$-\frac{\epsilon\mu}{4\pi c} \cdot \int \frac{\partial \dot{\phi}'}{\partial x'} \cdot \frac{d\tau'}{r} = \frac{\epsilon\mu}{4\pi c} \int \dot{\phi}' \frac{\partial \frac{1}{r}}{\partial x'} d\tau' \quad . \quad . \quad (360)$$

since when we integrate by parts with respect to x' the surface-integral vanishes because $\dot{\phi}$ and $\frac{1}{r}$ are both continuous. If we now consider that:

$$\frac{\partial \frac{1}{r}}{\partial x'} = -\frac{\partial \frac{1}{r}}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (361)$$

and that $\dot{\phi}'$ and $d\tau'$ do not depend on x, the expression (360) may also be written:

$$-\frac{\mu}{c} \frac{\partial \dot{\psi}}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (362)$$

where the scalar quantity:

$$\psi = \frac{\epsilon}{4\pi} \int \frac{\phi'}{r} d\tau' \quad . \quad . \quad . \quad . \quad . \quad . \quad (363)$$

and the vector-potential (359) of the whole current becomes:

$$A^* = A - \frac{\mu}{c}\operatorname{grad}\psi \quad . \quad . \quad . \quad . \quad (364)$$

Using (284) we obtain from this the important result:

$$H = \frac{1}{\mu}\operatorname{curl} A^* = \frac{1}{\mu}\operatorname{curl} A \quad . \quad . \quad . \quad (365)$$

Hence even in the case of unclosed currents the magnetic intensity of field is calculated from the vector-potential of the conduction currents without regard to the displacement currents although div A does not vanish.

§ 85. The position is different, however, in the case of the magnetic energy, as we shall immediately see. It is true that on account of (365) we also obtain for it the expression (332a), namely:

$$\frac{1}{8\pi}\int A \cdot \operatorname{curl} H \cdot d\tau$$

but the value (357) must now be substituted for curl H and this leads to the following amount for the required magnetic energy:

$$\frac{1}{2c}\int A \cdot \left(J + \frac{\dot{D}}{4\pi}\right) d\tau$$

which, by (289), we may also write in the form:

$$\frac{\mu}{2c^2}\int \frac{J'}{r}\left(J + \frac{\dot{D}}{4\pi}\right) d\tau d\tau'$$

or, if we interchange the accented and the unaccented quantities and introduce the vector-potential A^* of the whole current-flow, as:

$$\frac{1}{2c}\int J \cdot A^* d\tau$$

and lastly by (364), if we introduce (319), as:

$$= W - \frac{\mu}{2c^2}\int J \cdot \operatorname{grad}\psi \cdot d\tau \quad . \quad . \quad . \quad . \quad (366)$$

where W again denotes the electrodynamic potential of the conduction currents.

Hence in the case of unclosed currents the expression $-W$ of the magnetic energy which is valid for closed currents becomes supplemented by another term and to calculate this term completely we have to determine the value of the scalar function ψ. This can be done most directly by using the identity:

$$\Delta r = \frac{2}{r} \quad . \quad . \quad . \quad . \quad . \quad (367)$$

and by writing equation (363) in the form:

$$\psi = \frac{\epsilon}{8\pi}\int \dot{\phi}'\left(\frac{\partial^2 r}{\partial x'^2} + \frac{\partial^2 r}{\partial y'^2} + \frac{\partial^2 r}{\partial z'^2}\right) d\tau'$$

Integrating by parts we get, since $\dot{\phi}$ is continuous:

$$\psi = -\frac{\epsilon}{8\pi}\int\left(\frac{\partial\dot{\phi}'}{\partial x'}\cdot\frac{\partial r}{\partial x'} + \quad . \quad . \quad . \quad\right) d\tau'$$

$$= \frac{1}{8\pi}\int\left(\dot{D}'_x\frac{\partial r}{\partial x'} + \dot{D}'_y\frac{\partial r}{\partial y'} + \dot{D}'_z\frac{\partial r}{\partial z'}\right) d\tau'$$

Integrating once again we get, since div $\boldsymbol{D} = 0$:

$$\psi = -\frac{1}{8\pi}\int\left(\dot{D}'_{\nu'} + \dot{D}_{\nu}\right) r d\sigma'$$

where the integral is to be taken over the surfaces of all the conductors; or by (41) and (354):

$$\psi = -\frac{1}{2}\int \dot{h}' r d\sigma' = \frac{1}{2}\int J'_{\nu} r d\sigma' \quad . \quad . \quad (368)$$

We see that ψ vanishes for closed currents, since then $J_{\nu} = 0$ at the surfaces of all the conductors.

Now that we have found the value of ψ we can express the additional term, which occurs in (360) for the magnetic energy and which is characteristic for unclosed currents, either in the form of a surface integral to be taken over the ends of the conduction currents or in the form of a volume integral to be taken over all the conduction currents.

For the first purpose, remembering that div $\boldsymbol{J} = 0$ and $\boldsymbol{J}_\nu$ vanishes towards the insulator, we can write the additional term in the form:

$$\frac{\mu}{2c^2}\int \boldsymbol{J}_\nu \,.\, \psi \,.\, d\sigma \quad . \quad . \quad . \quad . \quad . \qquad (369)$$

where the integral is to be taken over the surfaces of all the conductors and so, by (368), we obtain the expression:

$$\frac{\mu}{4c^2}\iint \boldsymbol{J}_\nu \,.\, \boldsymbol{J}'_{\nu'} \,.\, r \,.\, d\sigma d\sigma' \quad . \quad . \quad . \quad . \qquad (370)$$

According to this every combination of two current ends makes a contribution to the magnetic energy. But this contribution is the smaller, the nearer the current ends lie together. This is easy to understand if we realize that the contribution is due to the displacement currents which flow between the ends of the currents and which in a certain sense form the continuation of the conduction currents, since they supplement the latter to form closed currents. For example, in the case of the discharge-current of an electrical condenser of very small thickness, the above contribution is vanishingly small and the magnetic energy is very nearly as great as if the current were closed.

For the second purpose we write the function ψ in the form:

$$\begin{aligned}\psi &= -\frac{1}{2}\int \operatorname{div}(\boldsymbol{J}'r) \,.\, d\tau' \\ &= -\frac{1}{2}\int\left(\boldsymbol{J}'_{x'}\frac{\partial r}{\partial x'} + \boldsymbol{J}'_{y'}\frac{\partial r}{\partial y'} + \boldsymbol{J}'_{z'}\frac{\partial r}{\partial z'}\right)d\tau' \\ &= -\frac{1}{2}\int |\,\boldsymbol{J}'\,| \cdot \frac{\partial r}{\partial s'} \cdot d\tau',\end{aligned}$$

where ds' denotes an element of a stream line. Further, we write the supplementary term in (366) in the corresponding form:

$$-\frac{\mu}{2c^2}\int |\,\boldsymbol{J}\,| \frac{\partial \psi}{\partial s} \cdot d\tau$$

and so arrive at the expression:

$$\frac{\mu}{4c^2}\iint |\boldsymbol{J}|\cdot|\boldsymbol{J}'|\cdot\frac{\partial^2 r}{\partial s\partial s'}d\tau d\tau' \quad . \quad . \quad (371)$$

All in all we then obtain for the magnetic energy of a system of unclosed conduction currents, by (366) and (320):

$$\frac{1}{2}\frac{\mu}{c^2}\iint |\boldsymbol{J}|\cdot|\boldsymbol{J}'|\left(\frac{\cos\delta}{r}+\frac{1}{2}\frac{\partial^2 r}{\partial s\partial s'}\right)d\tau d\tau' \quad . \quad . \quad (372)$$

The fact that the second term of the summand can be neglected can be seen at once if, by (291a) we write:

$$|\boldsymbol{J}|\cdot d\tau = J ds$$

and now perform the integrations over the line-elements ds and ds' of the closed current-filaments.

Instead of the differential coefficients of r with respect to s and s' we can also introduce besides the angle δ between the two line-elements also the angle which the direction of r makes with the directions of ds and ds' by means of the identity:

$$\frac{\partial^2 r}{\partial s\partial s'} = -\frac{\cos\delta}{r} - \frac{1}{r}\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} \quad . \quad . \quad . \quad (373)$$

which results when we differentiate the equation:

$$\frac{\partial r}{\partial s} = \frac{x-x'}{r}\frac{\partial x}{\partial s} + \frac{y-y'}{r}\frac{\partial y}{\partial s} + \frac{z-z'}{r}\frac{\partial z}{\partial s}$$

with respect to s'.

Summarizing the results of our calculations we may state the theorem that, so far as quasi-stationary processes come into question, the magnetic energy of a system of unclosed currents J_1, J_2, . . . may again be represented by the formula (324) deduced for closed currents with the modification that only the coefficients L become generalized in the manner of (372). In most cases this modification is of no account practically.

Having obtained the expressions for the electric and

the magnetic energy we can now make the same deductions as in the preceding chapter for closed currents.

§ 86. We shall now take as an example the important case of the discharge of an electric condenser. In the initial state ($t = 0$) let the charges on the two plates be $+e_0$ and $-e_0$, and let the intensity of the discharge current J, which by (355) is determined by the variation of the charges in time, be zero initially. The course of the process is then given uniquely by the application of the energy principle. Since the conductor is at rest and since no chemical changes occur we need take into account only the electric energy, the magnetic energy and the generation of heat.

We obtain the value of the electric energy from § 82 in the electrostatic portion of the present volume by applying the formula (221):

$$\frac{1}{2}e\phi = \frac{1}{2}\frac{e^2}{C} \quad . \quad . \quad . \quad . \quad . \quad (374)$$

where $C = \dfrac{e}{\phi}$ denotes the capacity of the condenser. By § 85 (concluding remarks) the magnetic energy is again determined by (334), the coefficient of self-induction L being constant in time here, and the heat generated in the time dt is given by (304).

The principle of the conservation of energy then leads to:

$$d\left(\frac{1}{2}\frac{e^2}{C}\right) + d\left(\frac{1}{2c^2}LJ^2\right) + J^2 w dt = 0$$

or, if (355) is taken into account:

$$\frac{L}{c^2}\frac{d^2e}{dt^2} + w\frac{de}{dt} + \frac{e}{C} = 0 \quad . \quad . \quad . \quad (375)$$

an equation which differs from the corresponding equation (336) for a closed current only in that here the potential difference at the two ends of the current, that is, at the plates of the condenser, becomes added. Comparison with I (19a) shows that it agrees exactly with

the equation which represents the motion of a uniformly damped oscillator, the coefficient of self-induction L playing the part of the inert mass, the reciprocal of the electrostatic capacity C that of the attracting force and the galvanic resistance w that of the damping agent.

Hence the process of discharge has a different character according as the quantity:

$$w^2 - \frac{4L}{c^2C} \gtreqless 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (376)$$

In the first case, including the limiting (second) case, the discharge occurs from beginning to end in a definite direction; in the last case, if the resistance is sufficiently small, the process is oscillatory with a damped period, the frequency being:

$$\nu = \frac{c}{2\pi}\sqrt{\frac{1}{CL} - \frac{w^2c^2}{4L^2}} \quad . \quad . \quad . \quad . \quad . \quad (377)$$

which is independent of the initial charge e_0, the logarithmic decrement being:

$$\sigma = \frac{c^2w}{2L\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (378)$$

In all cases the original electric energy is finally to be found as Joule energy in the circuit and the path of the spark. For not only the resistance of the conducting substance but also the spark gap contributes to the conduction resistance w. The smaller the conduction resistance w, the slower the decrease in the amplitude of vibration and the longer the time taken for the oscillatory process to come to an end. In the case of weakly damped oscillations the frequency is very nearly, by (377):

$$\nu = \frac{c}{2\pi\sqrt{CL}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (379)$$

which is Kelvin's formula.

The oscillatory discharge was first subjected to measurement by W. Feddersen, who separated out spatially the sparks emitted successively in time by the discharge.

He achieved this by means of a rapidly rotating mirror. The frequency ν used by him, which is to be calculated from (379), was of the order of 10^6 per second. H. Hertz was the first to increase the frequency considerably by reducing the capacity and the self-induction. But we must leave the discussion of Hertzian waves till the next chapter because the processes effected by them in the electromagnetic field occur too rapidly to allow them to be regarded as quasi-stationary.

CHAPTER III

DYNAMICAL PROCESSES IN STATIONARY BODIES

§ 87. If an electromagnetic field changes so rapidly in space and in time that it can no longer be regarded as quasi-stationary all the methods hitherto used by us break down and there is nothing for us to do but to return to the general equations established in the second chapter of the first part of the present volume, and to deduce from them the relationships which correspond to the physical case to be investigated.

We shall first treat these equations for homogeneous isotropic bodies at rest and shall simplify them by showing that all the vectors that occur in them can be referred to one single function. We first solve the equation (51) for the magnetic induction $\boldsymbol{B}$ by generalizing (282) and setting:

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (380)$$

The electric intensity of field $\boldsymbol{E}$ satisfies the differential equation (31*b*):

$$\operatorname{curl} \dot{\boldsymbol{A}} = -c \operatorname{curl} \boldsymbol{E}$$

whose integral is:

$$\boldsymbol{E} = -\frac{\dot{\boldsymbol{A}}}{c} - \operatorname{grad} \phi \quad . \quad . \quad . \quad . \quad . \quad (381)$$

These results refer the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ and, by (28), (29), (30), also the vectors $\boldsymbol{D}$, $\boldsymbol{J}$ and $\boldsymbol{H}$ to the vector-potential $\boldsymbol{A}$ and the scalar potential ϕ.

It is possible to introduce a certain relationship arbitrarily between the two potentials $\boldsymbol{A}$ and ϕ without restricting the generality of the electromagnetic process.

For if we add the term grad ψ to the vector A and the term $-\frac{\dot{\psi}}{c}$ to the potential ϕ, where ψ denotes any scalar function of space and time, then by (380) $\boldsymbol{B}$ remains unchanged and also, by (381) $\boldsymbol{E}$.

The function ψ is quite arbitrary; it cannot be determined from the remaining field-equations, because they contain only the field-strengths and not the potentials.

We shall use the indefiniteness mentioned to resolve the differential equation resulting from (31a) into two simpler equations. This equation runs, in consequence of (286):

$$-\frac{\epsilon\ddot{A}}{c} - \epsilon \operatorname{grad} \dot{\phi} = \frac{c}{\mu} \operatorname{grad} \operatorname{div} A - \frac{c}{\mu}\Delta A + \frac{4\pi\kappa}{c}\dot{A} + 4\pi\kappa \operatorname{grad} \phi.$$

It is satisfied if we apply the result of our above remarks by setting the terms following the symbol grad equal to zero:

$$-\epsilon\dot{\phi} = \frac{c}{\mu}\operatorname{div} A + 4\pi\kappa\phi \quad . \quad . \quad . \quad . \quad (382)$$

and hence also:

$$-\frac{\epsilon\ddot{A}}{c} = -\frac{c}{\mu}\Delta A + \frac{4\pi\kappa}{c}\dot{A} \quad . \quad . \quad . \quad (383)$$

or, if we introduce the quantity:

$$\frac{c}{\sqrt{\epsilon\mu}} = q \quad . \quad . \quad . \quad . \quad . \quad . \quad (384)$$

which denotes a velocity, and the time of relaxation T from (21), we get:

$$\dot{\phi} + \frac{\phi}{T} = -\frac{q^2}{c}\operatorname{div} A \quad . \quad . \quad . \quad . \quad (385)$$

and:

$$\ddot{A} + \frac{\dot{A}}{T} = q^2\Delta A \quad . \quad . \quad . \quad . \quad . \quad (386)$$

The last differential equation, which is satisfied by the

vector potential A, is often called the "equation of telegraphy."

If we also take (50) into account or:

$$\frac{\operatorname{div} \dot{A}}{c} + \Delta\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad (387)$$

we obtain from (385) by differentiating with respect to t:

$$\ddot{\phi} + \frac{\dot{\phi}}{T} = q^2\Delta\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (388)$$

Hence also the scalar potential and consequently all the field-vectors satisfy the equation of telegraphy.

Any two expressions for the potentials A and ϕ—which both satisfy the equation of telegraphy and are also related as in (385)—represent a possible electromagnetic process.

Finally we can also refer the functions A and ϕ to a single vectorial function Z which need only satisfy the equation of telegraphy:

$$\ddot{Z} + \frac{\dot{Z}}{T} = q^2\Delta Z \quad . \quad . \quad . \quad . \quad . \quad . \quad (389)$$

while we satisfy (385) by setting:

$$A = \frac{c}{q^2}\left(\dot{Z} + \frac{Z}{T}\right) \quad . \quad . \quad . \quad . \quad . \quad (390)$$

and:

$$\phi = -\operatorname{div} Z \quad . \quad . \quad . \quad . \quad . \quad . \quad (391)$$

All the field-vectors then result uniquely from the one vector Z called the Hertzian vector, namely, by (381):

$$E = \operatorname{grad} \operatorname{div} Z - \Delta Z = \operatorname{curl} \operatorname{curl} Z \quad . \quad . \quad (392)$$

and by (380):

$$B = \frac{c}{q^2} \operatorname{curl}\left(\dot{Z} + \frac{Z}{T}\right) \quad . \quad . \quad . \quad . \quad . \quad (393)$$

For non-conducting media ($T = \infty$) we have, in particular:

$$B = \frac{c}{q^2} \operatorname{curl} \dot{Z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (394)$$

while the equation of telegraphy (389) becomes:

$$\ddot{Z} = q^2 \,.\, \Delta Z \quad . \quad . \quad . \quad . \quad . \quad . \quad (395)$$

the well-known wave-equation II (222), with the velocity of propagation q.

For a vacuum or a sufficiently rarefied gas ($\epsilon = 1$, $\mu = 1$) q merges into the critical velocity c and in this way the critical velocity acquires a direct physical meaning: it is the velocity of propagation of electromagnetic waves in a pure vacuum—a theorem which, on account of the fact that the critical velocity has the same value as the velocity of light, formed the starting-point of Maxwell's electromagnetic theory of light.

§ 88. The general solution is less interesting from the point of view of physical applications than a particular solution of the field-equations which is adapted to the physical conditions under discussion. We shall begin with this case which, as we shall see, gives us the theory of the rapid Hertzian vibrations mentioned at the end of the preceding chapter.

The wave-equation (395) for $q = c$ is satisfied, according to II (230), by the components of the vector $\boldsymbol{Z}$:

$$\boldsymbol{Z}_x = 0, \quad \boldsymbol{Z}_y = 0, \quad \boldsymbol{Z}_z = \frac{1}{r} \cdot f\left(t - \frac{r}{c}\right) \qquad (396)$$

where f denotes an arbitrary function of a single argument. By (392) and (394) we get from this the following expressions for the electric and magnetic components of the field:

$$\boldsymbol{E}_x = \frac{\partial^2 \boldsymbol{Z}_z}{\partial x \partial z} \qquad (397a)$$

$$\boldsymbol{E}_y = \frac{\partial^2 \boldsymbol{Z}_z}{\partial y \partial z} \qquad (397b)$$

$$\boldsymbol{E}_z = -\frac{\partial^2 \boldsymbol{Z}_z}{\partial x^2} - \frac{\partial^2 \boldsymbol{Z}_z}{\partial y^2} = \frac{\partial^2 \boldsymbol{Z}_z}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{Z}_z}{\partial t^2} \qquad (397c)$$

$$\boldsymbol{H}_x = \frac{1}{c}\frac{\partial^2 \boldsymbol{Z}_z}{\partial y \partial t}, \quad \boldsymbol{H}_y = -\frac{1}{c}\frac{\partial^2 \boldsymbol{Z}_z}{\partial x \partial t}, \quad \boldsymbol{H}_z = 0 \qquad (398)$$

We can assume that these expressions are valid in the whole of space except a small portion around the origin

$r = 0$, because here the Hertzian vector becomes infinitely great.

Let us now investigate the physical significance of the field-components in the neighbourhood of the origin. We can then write:

$$f\left(t - \frac{r}{c}\right) = f(t) - \frac{r}{c} \cdot \dot{f}(t) + \dots \quad (399)$$

Thus if we assume:

$$r \ll \frac{c \cdot f(t)}{\dot{f}(t)} \quad (400)$$

which quantitatively specifies the condition implied in the term "in the neighbourhood of the origin," the Hertzian vector reduces, by (396), to:

$$Z_z = \frac{1}{r} f(t)$$

and from (397) and (398) we obtain the following values for the field-components:

$$E_x = \frac{\partial^2 \frac{1}{r}}{\partial x \partial z} \cdot f(t), \quad E_y = \frac{\partial^2 \frac{1}{r}}{\partial y \partial z} \cdot f(t), \quad E_z = \frac{\partial^2 \frac{1}{r}}{\partial z^2} \cdot f(t) \quad (401a)$$

$$H_x = \frac{1}{c} \frac{\partial \frac{1}{r}}{\partial y} \cdot \dot{f}(t), \quad H_y = -\frac{1}{c} \frac{\partial \frac{1}{r}}{\partial x} \cdot \dot{f}(t), \quad H_z = 0 \quad (401b)$$

or, written in another way:

$$E = -\operatorname{grad} \phi, \quad H = \operatorname{curl} A$$

where:

$$\phi = -\frac{\partial \frac{1}{r}}{\partial z} f(t) = \frac{z}{r^3} f(t) \quad (402)$$

$$A_x = 0, \quad A_y = 0, \quad A_z = \frac{1}{c} \frac{\dot{f}(t)}{r} \quad (403)$$

These expressions represent a quasi-stationary field. By (145) the electric field with the potential ϕ has its source in an electric dipole at the origin, its axis being in the direction of the z-axis and its moment being $f(t)$;

by (289) the magnetic field with the vector-potential A has its source in a current-element, situated at the origin, of an unclosed current flowing in the z-direction, while $\dot{f}(t)$, by (292), gives the product of the current-strength and the length of the element. This is precisely the current which causes the change in the charge of the electric dipole just mentioned.

Hence we are justified in drawing the following conclusion: an electric dipole lying along the z-axis at the origin and having the moment $f(t)$ which can vary arbitrarily with the time generates in its immediate neighbourhood, that is, at distances which are of the order of magnitude indicated in (400), a quasi-stationary field which is determined by its moment at the particular instant and its change of current (*Umladungsstrom*) at this instant. At greater distances r, however, the equations (401) lose their validity and they become replaced by the more general expressions (397) and (398) which represent the only extensions of the field-vectors which are continuous and compatible with the dynamical field-equations for any values of the distance r whatsoever.

Let us next consider the opposite limiting case: the values of the field-strengths at distances from the origin which are so great that:

$$r \gg \frac{c \cdot f(t)}{\dot{f}(t)} \quad . \quad . \quad . \quad . \quad . \quad (404)$$

The only terms that then remain in the differentiations in (397) and (398) are those which arise from the differentiation of the argument $\left(t - \frac{r}{c}\right)$ of f, and we obtain:

$$E_x = \frac{xz}{c^2 r^3} \ddot{f}\left(t - \frac{r}{c}\right), \quad E_y = \frac{yz}{c^2 r^3} \ddot{f}\left(t - \frac{r}{c}\right) \quad . \quad . \quad . \quad . \quad (405a)$$

$$E_z = -\frac{x^2 + y^2}{c^2 r^3} \ddot{f}\left(t - \frac{r}{c}\right)$$

$$H_x = -\frac{y}{c^2 r^2} \ddot{f}\left(t - \frac{r}{c}\right), \quad H_y = \frac{x}{c^2 r^2} \ddot{f}\left(t - \frac{r}{c}\right), \quad H_z = 0 \; . \quad (405b)$$

These equations represent a spherical wave which propagates itself outwards with the velocity c and which is moreover a transverse wave, for both the electric intensity of field $\boldsymbol{E}$ and also the magnetic intensity of field $\boldsymbol{H}$ are perpendicular to the radius r of the spherical wave, as is shown by the relationships:

$$xE_x + yE_y + zE_z = 0$$

and

$$xH_x + yH_y + zH_z = 0$$

$$E_xH_x + E_yH_y + E_zH_z = 0.$$

Both the electric and the magnetic lines of force are circles and, on account of $H_z = 0$, the magnetic lines of force are the cross-sections of the spherical wave $r =$ const. with the planes parallel to the xy-plane, but the electric lines of force are the great circles of the spherical wave which are perpendicular to the magnetic lines of force and pass through the z-axis. The directions $\boldsymbol{E}$, $\boldsymbol{H}$, r always form a right-handed system and, by (9), the direction $\boldsymbol{S}$ of the energy-flux coincides with the direction r.

As for the values of the field-intensities, the electric intensity of field is always equal in magnitude to the magnetic intensity of field and simultaneously changes its sign with it. This common value is:

$$\frac{\sin\theta}{c^2 r}\ddot{f}\left(t - \frac{r}{c}\right). \quad . \quad . \quad . \quad . \quad . \qquad (406)$$

where θ denotes the angle between the radius vector r and the z-axis. Hence the field-intensities are zero on the z-axis whereas in the xy-plane they are a maximum.

As we see, the electromagnetic processes at distances for which r has the values given by (404) are totally different from those in the neighbourhood (400) of the origin. For example, in the case of periodic vibrations $f(t)$ the field in the immediate neighbourhood of the vibrating centre is likewise perfectly periodic whereas at great distances a transport of energy takes place only in

the outward direction, nothing remaining of the periodicity except the periodic change in the strength of the energy-flux. The continuous transition from the quasi-stationary vibration to the progressive wave is effected by the processes that occur at distances r which are of the order of magnitude $\frac{cf(t)}{\dot{f}(t)}$ and which are of a more complex character. H. Hertz has given a calculation and physical description of them for the case of periodic vibrations in *Wiedemann's Annalen der Physik*, Vol. 36 (1889).

§ 89. If we now enquire into the meaning of the formulae which have been evolved for the processes that accompany the discharge of an electric condenser we must bear in mind that a charged condenser may be regarded as a single electric dipole only for such distances r as are great compared with all the linear dimensions of the condenser. Hence we must from the very outset restrict our attention to those points of the surrounding field which satisfy this condition. But since again, to justify our regarding the field as quasi-stationary, the condition (400) must be fulfilled, it follows *a fortiori* that our equations can be applied only to those discharge processes in which the linear dimensions of the apparatus used are small compared with the quantity:

$$\frac{c \cdot f(t)}{\dot{f}(t)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (407)$$

This imposes a certain upper limit to the rapidity with which the process can take place. For the more quickly the changes occur the more the length (407) decreases. On account of the factor c it is usually so considerable under ordinary circumstances that the required condition is sufficiently well fulfilled in many cases. For example, let us take the case of an oscillatory discharge of frequency ν (§ 86). For this (407) is of the order of magnitude:

$$\frac{c}{\nu} = \lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (407a)$$

where λ denotes the length of a periodic wave advancing in space with the velocity c and of frequency ν. For Feddersen's vibrations $\nu = 10^6$, that is, $\lambda = 3 \cdot 10^4$ cms. = 300 metres. The vibrations produced by Hertz were about 100 times more rapid, that is, $\nu = 10^8$ and $\lambda =$ 3 metres for them; these figures are still great enough to allow us to regard the Hertzian oscillator approximately as a dipole and the field in its neighbourhood as quasi-stationary. In his experiments Hertz used, among other devices, a rectilinear conductor which included a spark-gap with a sphere at each end which served as a capacity. When the two spheres had been provided with opposite charges of a certain amount an oscillatory discharge, which died away more or less rapidly, took place at the spark gap. On account of the very small amount of energy involved in this process the charges were renewed by a continuously working influence machine or an induction coil whose poles were connected with the two spheres. In this way the whole process was repeated many times, just as, in order to make more audible the weak tone of a bell whose vibrations die down rapidly, we give it a rapid succession of raps.

So soon as the distance r of the point of the field under consideration becomes of the order (407) or of that of the wave-length λ the equations of the quasi-stationary process lose their validity, and at distances which are great compared with λ only the progressive spherical wave remains. This, of course, also holds for the slowest vibrations, so that we can say: in principle *every* electromagnetic field that varies in time and that is not outwardly limited loses energy by radiation into infinite space; or, expressed in other words, in the mutual transformations of electric and magnetic energy which are associated with every electrical vibration the exchange never occurs without a loss; rather, a smaller or a greater amount of energy always escapes into outer space.

§ 90. It is easy to calculate the amount of energy transported outwards by the spherical waves (405) in any

interval of time. To do this most conveniently we use the value (406) for the electric or the magnetic field-strength respectively. By (5) and (26) the energy which flows outwards in the time dt through a surface-element $d\sigma$ of the spherical wave is then:

$$\frac{c}{4\pi} \cdot \frac{\sin^2\theta}{c^4 r^2} \cdot \ddot{f}^2\left(t - \frac{r}{c}\right) . d\sigma . dt \quad . \quad . \quad . \quad (408)$$

and, if we integrate over the whole surface of the spherical wave, we obtain for the total energy emitted in the element of time dt the positive amount:

$$\frac{2}{3c^3}\ddot{f}^2\left(t - \frac{r}{c}\right) . dt \quad . \quad . \quad . \quad . \quad (409)$$

Consequently we have for the mean value of the energy emitted per unit of time in a periodic or nearly periodic vibration:

$$\frac{2}{3c^3}\overline{\ddot{f}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (410)$$

which is independent of space and time.

According to the energy principle this continual loss of energy requires to be compensated for, if the vibration is to be completely periodic, by the continual supply of energy to the source of vibration since, if such a supply is not available, the vibrations will become damped: the latter damping is called "radiation damping" to distinguish it from the damping caused by the generation of the Joule heat, to which it becomes added.

Thus for a periodically vibrating electric dipole the moment $f(t)$ is of the form:

$$f(t) = l . e_0 . \cos 2\pi\nu t \quad . \quad . \quad . \quad . \quad (411)$$

where l denotes the constant distance between the two poles, and e_0 the initial charge on a pole. By (379) the frequency ν is to be calculated from the capacity C and the self-induction L. The energy E of the vibration is constant, and by (374) is equal to:

$$E = \frac{1}{2}\frac{e_0^2}{C} = \frac{2\pi^2\nu^2 L e_0^2}{c^2} \quad . \quad . \quad . \quad . \quad (412)$$

The energy radiated out in the time of a vibration is obtained by integrating (409) from t to $t + \frac{1}{\nu}$, which gives:

$$\frac{16\pi^4\nu^3}{3c^3} \cdot l^2 e_0{}^2 \quad . \quad . \quad . \quad . \quad . \quad (413)$$

that is, an expression which is inversely proportional to the third power of the wave-length λ.

If this amount of energy is not continually replaced the vibrations become damped, the decrement of the damped vibrations being determined by the decrease in the energy of vibration in the course of a vibration:

$$E_0 - E_1 = \frac{16\pi^4\nu^3}{3c^3} \cdot l^2 e_0{}^2$$

or, by (412):

$$\frac{E_0 - E_1}{E_0} = \frac{8\pi^2\nu l^2}{3cL} \quad . \quad . \quad . \quad . \quad . \quad (414)$$

If the damping is weak this expression is at the same time the logarithmic decrement of the energy, that is, the natural logarithm of the ratio $E_0 : E_1$. Since the energy is proportional to the square of the amplitude, it follows from this that the logarithmic decrement of damping of the amplitude σ is half the preceding value, that is:

$$\sigma = \frac{4\pi^2\nu l^2}{3cL} \quad . \quad . \quad . \quad . \quad . \quad . \quad (415)$$

or, by (379) and (407a):

$$\sigma = \frac{16\pi^4 C l^2}{3\lambda^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (415a)$$

In the Fedderson vibrations the radiation decrement σ is extremely small, but in the Hertzian vibrations it already attains to the order of magnitude of the damping effected by the resistance of the conductors, and in the case of still more rapid vibrations it alone plays the decisive part.

§ 91. The solution of the field-equations which was considered in § 88 and which referred to a vibrating electric dipole holds not only for an element of an oscillat-

ing conduction current but also for the rectilinear vibrations of an electrically insulated charged point-mass (ion or electron) about a position of equilibrium at which a charge of opposite sign is situated. For since by § 81 an electric convection current has the same magnetic properties as the conduction current defined by (353):

$$J \,.\, l = e \,.\, q = \dot{f}(t) \quad . \quad . \quad . \quad . \quad (416)$$

where e denotes the constant charge and q the variable velocity; the surrounding electromagnetic field is the same in both cases. Accordingly, by (410), the energy emitted per unit of time by an oscillating ion is:

$$\frac{2}{3}\frac{e^2}{c^3} \cdot \overline{q^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (417)$$

To calculate the radiation damping from this we must bear in mind that the energy of the vibration in question is in general both mechanical (kinetic and potential) and electromagnetic. The former has its seat in the moving inertial mass and in the elastic force that acts on it, the latter in the surrounding electromagnetic field. In passing through the position of equilibrium, for which by our assumption the electric field vanishes, the velocity q assumes its maximum value $q_{max.}$; then there is only kinetic and magnetic energy. Since both of these are proportional to the square of the velocity—this is so in the case of the magnetic energy on account of (416)—they may be combined into a single term:

$$E = \frac{1}{2} m q^2_{max.} \quad . \quad . \quad . \quad . \quad . \quad (418)$$

where the constant factor m is called the "effective" inertial mass of the ion. It consists of a mechanical component, which is directly given, and an electrical component which is to be calculated from the magnetic field of the moving ion. An electron possesses *only* electromagnetic mass.

If ν again denotes the frequency, then q is of the form:

$$q = q_{max.} \,.\, \cos\,(2\pi\nu t + \theta)$$

and :

$$\overline{\dot{q}^2} = 2\pi^2\nu^2 q^2_{\text{max.}} \quad . \quad . \quad . \quad . \quad . \quad (419)$$

and hence the logarithmic decrement of the energy is, by (417) and (418) :

$$\frac{E_0 - E_1}{E_0} = \frac{8\pi^2 e^2 \nu}{3c^3 m}$$

and the logarithmic decrement of the amplitude is half this quantity, namely :

$$\sigma = \frac{4\pi^2 e^2 \nu}{3c^3 m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (420)$$

If m is entirely electromagnetic by nature, comparison with (415) gives :

$$m = \frac{e^2 L}{c^2 l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (421)$$

for the relationship between the electromagnetic mass of a moving electric charge e for which its electromagnetically equivalent current-element of length l has the coefficient of self-induction L.

This can also be deduced directly by comparing the two expressions for the magnetic energy :

$$\frac{1}{2}\frac{L}{c^2} J^2 = \frac{1}{2} m q^2$$

in accordance with (416).

In physical nature it is not possible to exhibit an essential difference between mechanical and electromagnetic inertia, so it appears justifiable to make the simplifying hypothesis that every kind of inertia is of electromagnetic origin.

§ 92. To test the form of the function $f(t)$, which is characteristic for the spherical wave (405) and, in particular, to investigate its periodicity the simplest method is to allow the wave to be reflected by a mirror. For in the case of a singly periodic wave stationary vibrations are then formed (II, § 40) and the periodicity is spread out in space in the nodes and loops, whereas in the case of a progressive

wave no point in space is favoured beyond the rest. Since it is not necessary to use the whole spherical wave in such an investigation it is advantageous to select from the spherical wave a portion which is so small compared with the radius r that the wave can be treated as plane in this part. Nevertheless, on account of (404) the dimensions of this wave-plane can still be great compared with the wave-length λ.

We derive the laws for the propagation of a plane wave most simply and directly from the equations (392) and (394), by using a particular solution of (395) for which again $\mathbf{Z}_x = 0$ and $\mathbf{Z}_y = 0$, say, while $\mathbf{Z}_z$ depends only on x and t. If we again assume $q = c$ this leads to:

$$\frac{\partial^2 \mathbf{Z}_z}{\partial t^2} = c^2 \frac{\partial^2 \mathbf{Z}_z}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \qquad (422)$$

and by II (157) the general integral of this equation is:

$$\mathbf{Z}_z = F\left(t + \frac{x}{c}\right) + G\left(t - \frac{x}{c}\right) \quad . \quad . \quad . \qquad (423)$$

This gives the following values for the field-components:

$$\mathbf{E}_x = 0,\ \mathbf{E}_y = 0,\ \mathbf{E}_z = f\left(t + \frac{x}{c}\right) + g\left(t - \frac{x}{c}\right) \qquad (424a)$$

$$\mathbf{H}_x = 0,\ \mathbf{H}_y = f\left(t + \frac{x}{c}\right) - g\left(t - \frac{x}{c}\right),\ \mathbf{H}_z = 0 \qquad (424b)$$

where we have used the abbreviations:

$$-\frac{1}{c^2}\ddot{F} = f, \quad -\frac{1}{c^2}\ddot{G} = g.$$

Hence we here have two waves which are in general independent of each other and which propagate themselves in space with the velocity c in the directions of the positive and the negative x-axis.

We shall now assume space to be limited by the plane $x = 0$, so that it is restricted to positive values of x. Let the space corresponding to negative values of x be filled by a substance of infinitely great conductivity κ, a so-

called "absolute" or "perfect" conductor, which is represented to a sufficient degree of approximation by practically all metals. Then, as we shall see, the plane $x = 0$ acts like a perfectly reflecting mirror towards an electromagnetic wave that falls on it from air space.[1]

For in a perfect conductor, since the current density $J = \kappa E$ cannot possibly become infinitely great, the electric intensity of field E and hence also the Joule heat κE^2 are necessarily zero everywhere, and since according to the general electromagnetic boundary conditions (11) the field-component E_z at the boundary $x = 0$ has the same value in both media, E_z is also always equal to zero in air space for $x = 0$, and so by (424a):

$$f(t) + g(t) = 0$$

is valid for all positive and negative values of the argument t. From this it also immediately follows that:

$$g\left(t - \frac{x}{c}\right) = -f\left(t - \frac{x}{c}\right)$$

and hence the field-equations (424) become:

$$E_z = f\left(t + \frac{x}{c}\right) - f\left(t - \frac{x}{c}\right) \quad . \quad . \quad . \quad (425a)$$

$$H_y = f\left(t + \frac{x}{c}\right) + f\left(t - \frac{x}{c}\right) \quad . \quad . \quad . \quad (425b)$$

If a wave $f\left(t + \frac{x}{c}\right)$ of arbitrary form which comes out of air space and advances in the negative x-direction falls on the surface $x = 0$ of the conductor, it is completely reflected in the opposite direction, the electric intensity of field with a negative and the magnetic intensity of field with a positive sign.

If the wave is singly periodic:

$$f(t) = a \cos(\omega t + \theta) \quad . \quad . \quad . \quad . \quad (426)$$

[1] The reason for introducing this term will be seen later; cf. p. 231.

and from (425) we get the following expressions for the field-strengths:

$$E_z = -2a \sin \frac{\omega x}{c} \sin(\omega t + \theta) \quad . \quad . \quad (427)$$

$$H_y = 2a \cos \frac{\omega x}{c} \cos(\omega t + \theta) \quad . \quad . \quad . \quad (428)$$

They represent a stationary wave with equidistant nodes and loops. Since the length of the progressive wave is $\frac{2\pi c}{\omega} = \lambda$, the nodes of the electric intensity of field are at:

$$x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots$$

and those of the magnetic intensity of field are at:

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots$$

Since the first waves produced by Hertz were over 1 metre long it was easy to separate the nodes and antinodes (or loops) in them.

§ 93. There is one point which is a little unsatisfactory in the above method of deriving the laws of reflection of a periodic wave at the surface of a perfect conductor. It is the circumstance that the tangential component of the magnetic intensity of field H_y is continuous at the bounding surface $x = 0$ of the perfect conductor according to the general boundary condition (11) and that hence it does not vanish at this point in the conductor nor in air space but rather, by (428), has a maximum amplitude, and so magnetic vibrations take place even inside the conductor whereas on the other hand the electric intensity of field is everywhere zero there according to our assumption.

We can shed some light on this apparent contradiction if we assume the conductivity κ of the conductor to be *not* infinitely great. We then have the more general case where the incident wave is not completely reflected

but penetrates partly into the conducting substance. Hence we must also set up the field-equations for the latter, that is, for $x < 0$. For simplicity we shall assume for the conductor as for air space that $\epsilon = 1$, $\mu = 1$. By (21) and (384) the equation of telegraphy then reduces to:

$$\ddot{\mathbf{Z}} + 4\pi\kappa\dot{\mathbf{Z}} = c^2\Delta\mathbf{Z}$$

or, if we again assume that $\mathbf{Z}_x = 0$, $\mathbf{Z}_y = 0$, and that $\mathbf{Z}_z$ depends only on x and t:

$$\ddot{\mathbf{Z}}_z + 4\pi\kappa\dot{\mathbf{Z}}_z = c^2\frac{\partial^2 \mathbf{Z}_z}{\partial x^2} \quad . \quad . \quad . \quad . \quad (429)$$

This differential equation can be integrated by means of an exponential function. For if we set:

$$\mathbf{Z}_z = e^{\omega\left(it + \frac{p}{c}x\right)} \quad . \quad . \quad . \quad . \quad . \quad (430)$$

where we assume the constant ω, the frequency, to be real, since we are dealing with a process which is periodic in time, whereas the constant p can be complex. The equation (429) is then satisfied if we set:

$$-\omega^2 + 4\pi\kappa \, . \, i\omega = \omega^2 p^2 \quad . \quad . \quad . \quad (431)$$

We shall now introduce a considerable simplification which is permissible for our purpose; it consists in assuming κ to be not infinitely great but only great compared with the frequency ω. The last equation then becomes:

$$p = \sqrt{\frac{4\pi\kappa i}{\omega}} = \sqrt{\frac{2\pi\kappa}{\omega}}\,(1 + i) = \sqrt{\frac{4\pi\kappa}{\omega}} \cdot e^{\frac{\pi}{4}i} \quad . \quad (432)$$

The root is to be taken positive since $\mathbf{Z}_z$ cannot become infinite for $x = -\infty$. For the field-intensities in the conductor we get from (302) and (393), inserting a complex constant C:

$$\mathbf{E}_z = C \, . \, e^{\omega\left(it + \frac{p}{c}x\right)} \quad . \quad . \quad . \quad . \quad (433)$$

$$\mathbf{H}_y = -\,ipC \, . \, e^{\omega\left(it + \frac{p}{c}x\right)} \quad . \quad . \quad . \quad (434)$$

which can be more simply derived directly from the field-equation:

$$\frac{\partial H_y}{\partial t} = c\frac{\partial E_z}{\partial x}.$$

On the other hand we correspondingly obtain for the field-intensities in space the following particular solution from (424):

$$E_z = Ae^{\omega i\left(t+\frac{x}{c}\right)} + Be^{\omega i\left(t-\frac{x}{c}\right)} \quad . \quad . \quad . \quad (435)$$

$$H_y = Ae^{\omega i\left(t+\frac{x}{c}\right)} - Be^{\omega i\left(t-\frac{x}{c}\right)} \quad . \quad . \quad . \quad (436)$$

At the boundary $x = 0$ E_z and H_y are continuous, hence:

$$C = A + B \text{ and } -ipC = A - B.$$

From this we get, if A is arbitrarily given:

$$B = \frac{1+ip}{1-ip}\cdot A = -\left(1-\sqrt{\frac{\omega}{2\pi\kappa}}(1+i)\right).A = -(1-\delta)e^{-\delta i}.A \quad (437)$$

$$C = \frac{2A}{1-ip} = \sqrt{\frac{\omega}{2\pi\kappa}}(1+i)\,A = \delta\sqrt{2}\cdot e^{\frac{\pi}{4}i}A \qquad (438)$$

in which we have set the small positive quantity:

$$\sqrt{\frac{\omega}{2\pi\kappa}} = \sqrt{\frac{\nu}{\kappa}} = \delta \quad . \quad . \quad . \quad . \quad (438a)$$

as an abbreviation.

We can now immediately derive a real solution of the problem from the complex solution so found if we recollect that both the field-equations and also the boundary conditions are linear and homogeneous with respect to the field-strengths. For since the above complex expressions satisfy all the conditions in the interior and at the boundary, so also do the real part and the purely imaginary part taken alone. Hence we need only take the real part of all the expressions to obtain a real solution which satisfies all the conditions of the problem. If we also set the initially arbitrarily given complex constant:

$$A = \alpha\cdot e^{\theta i} \quad . \quad . \quad . \quad . \quad . \quad (439)$$

then the following real quantities result from (435) and (436) in the manner given for air space:

$$E_z = a\cos\left\{\omega\left(t+\frac{x}{c}\right)+\theta\right\} - (1-\delta)a\cos\left\{\omega\left(t-\frac{x}{c}\right)-\delta+\theta\right\} \quad (440)$$

$$H_y = a\cos\left\{\omega\left(t+\frac{x}{c}\right)+\theta\right\} + (1-\delta)a\cos\left\{\omega\left(t-\frac{x}{c}\right)-\delta+\theta\right\} \quad (441)$$

and from (433) and (434) we get for the conductor:

$$E_z = \delta\sqrt{2}\cdot a\cdot e^{\frac{\omega x}{\delta c}}\cos\left\{\omega\left(t+\frac{x}{\delta c}\right)+\frac{\pi}{4}+\theta\right\} \quad . \quad (442)$$

$$H_y = 2a\cdot e^{\frac{\omega x}{\delta c}}\cos\left\{\omega\left(t+\frac{x}{\delta c}\right)+\theta\right\} \quad . \quad . \quad (443)$$

The fact that the preceding expressions actually satisfy all the conditions in the interior and at the boundary of the media can of course be verified subsequently by substitution in the field equations.

Thus our problem of determining the reflection of the incident wave (426) when the conductor has a finite conductivity which is great compared with the frequency ω is solved. For $\kappa = \infty$ or $\delta = 0$ we of course again arrive at the formulae (427) and (428) which hold for reflection from a perfect conductor. A finite κ, on the other hand, effects a slight change both in the amplitude and also in the phase of the reflected wave.

This also explains the peculiar circumstance, mentioned at the beginning of this section, that within the conductor finite magnetic vibrations occur whereas the electric field-strength in the conductor is vanishingly small. Actually, $\boldsymbol{E}$ is small compared with $\boldsymbol{H}$ everywhere in the conductor, and this is made possible by the fact that the field-strengths change very rapidly with the distance x from the boundary surface. For even at the distance of a single wave-length $x = \lambda$ the field-strength has dropped, on account of the great value of the exponent $\dfrac{\omega\lambda}{\delta c} = \dfrac{2\pi}{\delta}$, to a very small fraction of its value at the boundary.

The wave penetrates the more deeply into the conductor the smaller the frequency ω and the smaller the conductivity κ.

Let us now glance at the energy conditions, which offer themselves most readily to measurement. The amount of energy that falls in unit time on unit area of the boundary or that starts out from it is given by Poynting's theorem as:

$$\frac{c}{4\pi}\int_0^1 E_z H_y \cdot dt$$

if we set $x = 0$ in it.

This gives, if we disregard the common factor $\frac{c}{4\pi}$, $\frac{\alpha^2}{2}$ for the incident wave, $\frac{\alpha^2(1-\delta)^2}{2}$ for the reflected wave, $\alpha^2\delta$ for the wave that penetrates into the conductor. The algebraic sum of these three expressions vanishes, as it should do. The last of them is identical with the energy transformed into Joule heat inside the conductor.

The ratio of the reflected and the transmitted energy to the incident energy is called the reflective and the absorptive power, respectively, of the substance for normal incidence. By the above their values are $1 - 2\delta$ and 2δ. Accurate measurements by E. Hagen and H. Rubens (1903) for the case of infra-red waves of light have shown that the theory agrees exactly with experiment. The above theory loses its significance, however, for short wave-lengths because then the distribution of matter in the conductor may no longer be assumed to be continuous.

§ 94. An electromagnetic wave which is incident from air space into a reflecting or an absorbing body exerts a *mechanical* action on it. We shall calculate this effect for the case of normal reflection at a conductor, as above considered. For this purpose we shall consider a cylindrical portion of the conductor, whose base is unit area of the surface of the conductor ($x = 0$) and whose height is assumed to be so great that the field components vanish at the other end. Then, by § 44, mechanical

pressures of electrical origin act on the whole surface of this cylinder, and their resultant reduces to the pressure on the base, where, by (231), the pressure-components amount to:

$$\left.\begin{aligned} X_x = Y_y = \frac{E_z^2}{8\pi},\ Z_z = \frac{E_z^2}{8\pi} \\ X_y = Y_z = Z_x = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (444)$$

Now since the inward normal of the surface of the body is equal to $-x$, the required mechanical force becomes, by II. (74):

$$X_\nu = -X_x = -\frac{E_z^2}{8\pi},\ Y_\nu = 0,\ Z_\nu = 0.$$

By § 46, a mechanical force of magnetic origin is to be added to this, which correspondingly runs:

$$X_\nu = -\frac{H_y^2}{8\pi},\ Y_\nu = 0,\ Z_\nu = 0$$

so that the total mechanical force that acts on the cylinder assumes the value:

$$X_\nu = -\frac{E_z^2 + H_y^2}{8\pi},\ Y_\nu = 0,\ Z_\nu = 0 \quad . \quad . \quad (445)$$

It acts in the negative direction of the x-axis, from air space into the conductor, and this represents a *pressure*, which is called the "radiation pressure." It is equal in amount to the volume energy-density of the radiation present in air space, the incident and the reflected waves being added together. The less a substance reflects the smaller is the radiation pressure exerted on it by an incident wave.

§ 95. We shall close this chapter by considering a particular solution of the field-equations which corresponds to the case where the electromagnetic waves do not propagate themselves spherically in all directions of space but progress in a perfectly definite direction along parallel wires (Lecher's apparatus). For this purpose we integrate the wave-equation (395) with $q = c$ by setting:

$$\mathbf{E}_x = 0,\ \mathbf{E}_y = 0,\ \mathbf{E}_z = F\left(x,\ y,\ t - \frac{x}{c}\right). \quad . \quad (446)$$

where we assume that the variables z and t occur only in the combination $t - \frac{z}{c}$.

The wave-equation then reduces to:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad (447)$$

and, by (397) and (398) the field-components become:

$$\mathbf{E}_x = -\frac{1}{c}\frac{\partial^2 F}{\partial x \partial t},\ \mathbf{E}_y = -\frac{1}{c}\frac{\partial^2 F}{\partial y \partial t},\ \mathbf{E}_z = 0 \quad . \quad (448)$$

$$\mathbf{H}_x = \frac{1}{c}\frac{\partial^2 F}{\partial y \partial t},\ \mathbf{H}_y = -\frac{1}{c}\frac{\partial^2 F}{\partial x \partial t},\ \mathbf{H}_z = 0\ . \quad . \quad (449)$$

If for abbreviation we set:

$$\frac{1}{c}\frac{\partial F}{\partial t} = \phi \quad . \quad . \quad . \quad . \quad . \quad (450)$$

we have:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad (451)$$

and:

$$\mathbf{E}_x = -\frac{\partial \phi}{\partial x} = \mathbf{H}_y,\ \mathbf{E}_y = -\frac{\partial \phi}{\partial y} = -\mathbf{H}_x \quad . \quad (452)$$

This is exactly the same as the form of the equations for a stationary irrotational (*wirbelfrei*) motion of an incompressible liquid (II, § 66), with the velocity potential ϕ, in the xy-plane or in a plane parallel to it. The stream-lines correspond to the electric lines of force, the lines of constant velocity-potential, ϕ = const., correspond to the magnetic lines of force, which are perpendicular to the electric lines of force. But there is one fundamental difference in that here the function ϕ depends on the parameter $t - \frac{z}{c}$ and, moreover, in a perfectly arbitrary manner, so that the electromagnetic field for a definite z is not stationary but displaces itself unchanged in the direction of increasing z with the velocity c.

Let us now turn to the boundary conditions. If an

arbitrary number of cylindrical wires, which we shall regard as perfect conductors, are held stretched in air parallel to the z-axis then, according to the boundary conditions (§ 92) that hold for a perfect conductor the electric lines of force are everywhere parallel to the xy-plane and end perpendicularly to the cross-sections of the wires, whereas the magnetic lines of force consequently coincide at the boundary with the edges of the cross-sections. In the analogy of fluid motion every cross-section thus represents a plane "source" or "sink" according as the cross-section is positively or negatively charged. Of course, corresponding to every positive charge, which is the beginning of a line of force, there is an equally great negative charge, which is the end of the line of force. This does not exclude the possibility of lines of force running off to infinity, as, for example, when only one wire is present. We can then consider the whole wire to be enclosed in a conducting hollow cylinder of very great cross-section which becomes correspondingly charged.

If the plane field now displaces itself along the z-axis then, in a definite plane parallel to the xy-plane, the charges of the cross-sections and, correspondingly, the positions of the lines of force change in a manner which is determined by the dependence of the function ϕ on the argument $t-\frac{z}{c}$ and which ultimately depends on the form of the electric vibration which is transmitted into the system of conductors.

The field-strengths are always greatest in the neighbourhood of the wires. Hence the stream of electromagnetic energy travels mainly along the wires and we may say that a conductor which stretches to a great distance has the property of concentrating in its immediate vicinity a wave which is progressing in its direction. This also explains why wireless waves do not become lost in space but follow the surface-curvature of the conducting earth.

Instead of using the above analogy of flow we can also use, by a well-known theorem in hydrodynamics (II, § 66), a different model, which is to some extent the reverse, to picture the electromagnetic field, namely by interchanging the stream-lines and the equipotential lines. The equipotential lines then begin and end in a normal direction on the cross-sections of the wire while the stream-lines encircle the cross-sections, and the wires represent cylindrically shaped vortex filaments, in which, however, every cross-section has its own particular vortex-moment (*Wirbelmoment*) determined by the argument $t-\frac{z}{c}$. If, for example, we have only two parallel wires of infinitely small cross-section, we get the picture of two parallel vortex filaments. The electric lines of force (equipotential lines) in any plane perpendicular to them are the circles which pass through the two points of intersection with the wires, and the magnetic lines of force (the stream-lines) are the circles which are perpendicular to them and which cut the straight line joining the two points harmonically. Cf. II, § 76, where the formulae referring to this problem are derived. The charges on the wires change with the time but at every moment they are equal and opposite at opposite points on the wires.

To find the form of the wave, that is, the dependence of the wave-function F on the argument $t-\frac{z}{c}$ we can make use of the same device as for free waves, namely transform the progressive wave by complete reflection into a stationary wave. This is accomplished most simply by connecting the cross-sections of the wire in any arbitrarily chosen plane z by means of a perfect conductor, that is, by forming a bridge. Under the conditions that here obtain in this plane no electric line of force is possible, that is, $E=0$. The whole plane behaves as if it were an absolute conductor. This boundary condition is satisfied if we assume besides the wave F advancing in

the positive direction of z also a reflected wave G, and hence replace (446) by:

$$Z_t = F\left(x, y, t - \frac{z}{c}\right) + G\left(x, y, t + \frac{z}{c}\right) \quad . \quad (453)$$

By (397) and (398) the field-components then become:

$$E_x = -\frac{1}{c}\frac{\partial^2(F - G)}{\partial x \partial t},\ E_y = -\frac{1}{c}\frac{\partial^2(F - G)}{\partial y \partial t},\ E_z = \ldots \quad (454)$$

$$H_x = \frac{1}{c}\frac{\partial^2(F + G)}{\partial y \partial t},\ H_y = -\frac{1}{c}\frac{\partial^2(F + G)}{\partial x \partial t},\ H_z = 0 \quad (455)$$

If we set $z = 0$ for the bridge, the boundary condition runs:

$$F(x, y, t) - G(x, y, t) = 0$$

which holds for all values of x, y and t. From this it follows generally that:

$$G\left(x, y, t + \frac{z}{c}\right) = F\left(x, y, t + \frac{z}{c}\right)$$

and hence:

$$Z_z = F\left(x, y, t - \frac{z}{c}\right) + F\left(x, y, t + \frac{z}{c}\right) \quad . \quad (456)$$

If the wave F is periodic with respect to t, we get from this, as in § 92, a stationary wave with nodes and anti-nodes at regular distances. In the plane of the bridge and in every parallel plane whose distance from it is an integral multiple of $\frac{\lambda}{2}$ the electric force has a node and the magnetic force an anti-node. In such a plane there are no electric but only magnetic lines of force; these lines are circular and embrace both wires. The maximum fluctuations or the anti-nodes of the current flowing in the wires correspond to them. If we connect the two wires here by a conductor nothing is altered in the phenomenon.

Between every two adjacent nodal planes of the electric force there are the nodal planes of the magnetic force. Here the field is purely electric, the lines of force run

alternately from one wire to the other, according as the charge changes, while the current-strength in the wires is permanently zero, which is fully analogous to the state of affairs at the node of a stationary air-wave, where the density of the air fluctuates to and fro from one maximum to another while the velocity of the air is always zero. So soon as a sensitive conductor, such as a Geissler tube, is inserted in the circuit at this point it renders manifest the electric potential between the wires by some reaction such as glowing, and this causes a more or less considerable disturbance in the phenomenon.

CHAPTER IV

DYNAMICAL PROCESSES IN MOVING BODIES. LIMITS OF THE ELECTRODYNAMICS OF MAXWELL AND HERTZ

§ 96. Now that we have dealt with the different special applications of the theory it remains for us to establish the fundamental equations for arbitrary dynamical processes in bodies moving in any way whatsoever, which constitutes the highest and ultimate goal of the present introduction to electrodynamics. It is obvious that those equations must contain as special cases all the laws obtained earlier both for bodies at rest as well as for quasi-stationary processes in moving bodies. We shall also seize the present opportunity to introduce a generalization of another kind by taking into consideration not only, as hitherto, homogeneous and isotropic bodies, but also non-homogeneous and anisotropic substances. A further very particular advantage accrues from this, namely all the boundary conditions can be completely dispensed with, as the following simple argument shows.

So long as we restrict ourselves to homogeneous substances we have to assume an abrupt transition at the boundary of two substances and have consequently to consider the question of continuity for every quantity that comes into account, as we actually did everywhere in the preceding chapters. But as soon as the equations established are also valid for non-homogeneous substances it is permissible to assume that the transition from one homogeneous substance to another does not occur abruptly through a surface, but continuously through a volume layer of small but finite thickness, and every discontinuity

in the phenomena that occur in the substances vanishes together with the discontinuities in the substances. Hence we now consider all constants and variables as continuous in space and time. Consequently the concept of the density h of surface charge lapses, and also that, g, of the electrical double-layer. These quantities are replaced by the space charge-density k in the transitional layer between two substances, as was discussed, for example, on an earlier occasion in § 29.

There is, however, one important assumption which we *must* and *shall* maintain in the sequel if we are to be able to carry to its conclusion the theory here being developed. This assumption is that the space occupied by matter must everywhere be occupied by it perfectly continuously, otherwise we shall not be able to maintain the continuity of the processes in space. The velocity of moving masses is also to be assumed as continuous throughout; for example, whenever two bodies glide over each other along their surface of contact this motion can be replaced by a continuous transition of the tangential velocity through a thin boundary layer which then of course becomes correspondingly deformed.

This assumption also has a bearing on the fact that there can be no absolute boundary to a body and hence also no absolute vacuum. For at such a boundary the velocity of the moving mass would experience a sudden change or rather would suddenly lose all meaning. Instead, we must imagine that even the most highly evacuated spaces still contain matter, as is actually the case, and, further, that this matter, this remainder of gas, occupies space continuously and moves with a velocity which links up continuously with the velocity of the walls of the containing vessel. It is true that this last point, which is of essential importance for the electrodynamics of Maxwell and Hertz, has shown itself to be fateful for its further development, in view of the fact that such small vestiges of gas are of no importance for the propagation of electromagnetic waves (§ 7).

§ 97. We shall arrive at our goal most easily if we first seek that form of the field-equations, in which no quantities that refer to the special nature of the material substances, such as dielectric constant, conductivity and so forth, occur. This has already been done for stationary bodies in the equations (31), which contain the generally valid laws of field-vectors, whereas the more special laws, including those characteristic of non-homogeneity and anisotropy, appear to be transplanted into the individual relationships between field-intensity and induction or current-strength, respectively. The problem now is to generalize these laws, expressed in (31), to bodies moving in any way whatsoever by retaining the relationships, which hold for stationary bodies, for the individual relationships as expressed by (28), (29) and (30). We shall use as our guiding idea the theorem, which immediately suggests itself and which has been confirmed by the results obtained in the preceding chapter, that *in sufficiently small regions of space every process may be regarded as quasi-stationary*. To formulate this law we can, as always, use either the integral or the differential form according to our requirements. We shall first use the former here as it can be more easily visualized and more conveniently expressed. The combination of the relations (347) and (348) then at once gives us the relationship between magnetic induction and electric intensity of field for a small closed material curve :

$$\frac{d}{dt}\int \boldsymbol{B}_\nu d\sigma = -c \,.\oint \boldsymbol{E}\,.\,dr \quad . \quad . \quad . \quad (457)$$

that is, the rate of change of the magnetic flux of induction through a surface bounded by the curve, no matter whether the change is produced by the motion of the matter or by a local change of the magnetic field, is equal to the negative product of the critical velocity and the line-integral of the electric intensity of field or of the electric potential along the whole curve.

The form of this relation is such that it can also be enunciated without limitations for any arbitrary finite

closed material curve. For if we have any material surface σ bounded by such a curve and resolve it into a great number of small material parts of the surface then the relationship (457) holds for every single part and its boundary. If we now add together all these equations we immediately obtain on the left-hand side the flux of induction through the whole surface σ, and on the right-hand side the sum of the line-integrals reduces to a single line-integral which is to be taken along the boundary curve of σ, as is clear if we observe that the sum of the line-integrals contains every line-element $d\mathfrak{r}$ situated in the interior of σ, once with a positive and once with a negative sign because the boundary $d\mathfrak{r}$ is in each case to be taken in the sense corresponding to that of the surface normal ν.

Hence (457) holds for any arbitrary closed material curve. The shape of the surface σ bounded by the curve is a matter of indifference since the magnetic flux of induction does not depend on it.

We shall now pass straight on from the integral form of the relationships between magnetic induction and electric intensity of field to the differential form in order to be able to compare the result with the differential equation (31*b*) for stationary bodies.

For this purpose we shall first transform the two sides of the equation (457) into the form of a surface-integral. This is easily accomplished for the right-hand side by applying Stokes's Theorem:

$$-c \cdot \oint \boldsymbol{E} \cdot d\mathfrak{r} = -c \int (\operatorname{curl} \boldsymbol{E})_\nu \cdot d\sigma \quad . \quad . \quad (458)$$

whereas the time-differential on the left-hand side can be resolved into two parts in the manner already shown in § 79: one part is due to the local change of the magnetic field and is represented by (348*a*), the other part is caused by the motion of the material curve, the magnetic field being kept constant, and its value is given by (350), or, according to Stokes's Law, by:

$$dt \cdot \int (\operatorname{curl} [\boldsymbol{B}, \mathfrak{q}])_\nu \cdot d\sigma \quad . \quad . \quad . \quad . \quad (459)$$

If we now make the surface σ contract to a single surface-element and remember that the direction of the normal ν can be quite arbitrary, then we get from (457), using (458), (348a) and (459), the general relationship:

$$\dot{\boldsymbol{B}} + \text{curl}\,[\boldsymbol{B}, \boldsymbol{q}] = -c\,.\,\text{curl}\,\boldsymbol{E} \quad . \quad . \quad (460)$$

which represents the required differential form of the relationship between magnetic induction and electric intensity of field. For $\boldsymbol{q} = 0$ it passes over into the differential equation (31b) for stationary bodies, as it should do; in the general case it shows itself to be fully identical with the equation (351a) already derived in § 79.

§ 98. We have next to deal with the relation, which is in a certain sense analogous, between electric induction and magnetic intensity of field, that is, with the generalization of the differential equation (31a) for moving bodies. We again first enquire into the integral form of this relationship.

It might suggest itself to us at first sight that we have a complete analogy and that we might equate the rate of change of the electric flux of induction through a surface σ with the here positive value of the product of the critical velocity and the line-integral of the magnetic intensity of field along the boundary curve of σ. But this is not feasible, because, in contrast with magnetic induction, the electric flux of induction through a surface σ depends not only on the boundary curve but also on the position of the surface otherwise, so that the equivalence suggested would have no meaning. But it is just by taking into account this circumstance that we obtain the means for obtaining the necessary generalizations.

For let us consider the difference in the electric flux of induction between two surfaces σ and σ' with their normals ν and ν' in the same direction and with the same boundary curve:

$$\int \boldsymbol{D}_{\nu'}\,.\,d\sigma' - \int \boldsymbol{D}_{\nu}\,.\,d\sigma \quad . \quad . \quad . \quad . \quad (461)$$

Then we immediately see from Fig. 12 that this difference is equal to the total flux of induction outwards from the space bounded by the two surfaces, and hence by (42) is equal to 4π times the electric charge e enclosed in the space.

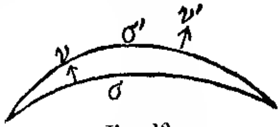

Fig. 12.

Now if the material points which constitute the two surfaces move in any way and if, besides, the electro-magnetic field changes in any way, it follows that:

$$\frac{d}{dt}\int D_{\nu'} \,.\, d\sigma' - \frac{d}{dt}\int D_\nu \,.\, d\sigma = 4\pi \,.\int \frac{de}{dt}$$

The quantity of electricity e enclosed in a space bounded by material surfaces can, however, change only if a conduction current passes into the interior through the surface of the space, and for the present case we have, by (43):

$$de = dt \,.\left(\int J_\nu d\sigma - \int J_{\nu'} \,.\, d\sigma'\right)$$

Consequently, by combining this with the previous equation and transposing terms, we have:

$$\frac{d}{dt}\int D_{\nu'} d\sigma' + 4\pi \,.\int J_{\nu'} d\sigma' = \frac{d}{dt}\int D_\nu d\sigma + 4\pi \int J_\nu d\sigma \,. \quad (462)$$

Hence we see that it is not the rate of change of the electric induction but this quantity increased by a term involving the conduction current that is entirely independent of the position of the surface σ for a definite boundary curve. This leads us to infer that the sum of these two quantities is an appropriate foundation for the relationship which we are seeking to establish. Hence we obtain the desired integral relationship in the form:

$$\frac{d}{dt}\int D_\nu d\sigma + 4\pi \,.\int J_\nu \,.\, d\sigma = c \,.\oint H \,.\, dr \quad . \quad (463)$$

The choice of the surface σ is quite immaterial. The only condition that must be satisfied is that in the differentiation with respect to the time on the left-band side

of the equation the surface σ must change in accordance with the motion of the material points that lie in it.

To follow up the integral form again with the differential form we adopt the same process as that used in the preceding paragraph in the case of magnetic induction. Firstly, we have on the right-hand side, just as in (458), by Stokes's Law:

$$c \, . \int (\text{curl}\, \boldsymbol{H})_\nu \, . \, d\sigma \quad . \quad . \quad . \quad . \quad (464)$$

On the left-hand side the expressions are a little more complicated than before. For in addition to the two terms into which the time differential:

$$d \int \boldsymbol{D}_\nu d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (465)$$

can be resolved, namely the one which is due to the local charge of the electric field;

$$dt \, . \int \dot{\boldsymbol{D}}_\nu \, . \, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (466)$$

and the other which is caused by the motion of the material boundary curve:

$$dt \, . \int (\text{curl}\, [\boldsymbol{D}, \boldsymbol{q}])_\nu \, . \, d\sigma \quad . \quad . \quad . \quad . \quad (467)$$

the electric field being kept constant, we now have a third term, namely, the change in the flux of induction through σ, which is produced when, with the electric field constant and the boundary curve fixed, the material points which form the surface σ are displaced. To calculate it we may again use Fig. 12 if we now take σ' to stand for the position of the surface at the time $t + dt$. The required quantity is then:

$$\int \boldsymbol{D}_\nu d\sigma' - \int \boldsymbol{D}_\nu d\sigma$$

or, written as a space-integral:

$$\int \text{div}\, \boldsymbol{D} \, . \, d\tau.$$

Here the space-element $d\tau$ is equal to $d\sigma \,.\, q_\nu \,.\, dt$, so that the expression is:

$$dt \,.\int \operatorname{div} \boldsymbol{D} \,.\, q_\nu \,.\, d\sigma \quad . \quad . \quad . \quad . \quad (468)$$

Hence the differential (465) is represented by the sum of (466), (467) and (468). If we now proceed as in the preceding section, the differential law for the relationship between the electric induction and magnetic field-strength follows from (463) in the form:

$$\dot{\boldsymbol{D}} + \operatorname{curl}\,[\boldsymbol{D}, \boldsymbol{q}] + \operatorname{div} \boldsymbol{D} \,.\, \boldsymbol{q} + 4\pi \boldsymbol{J} = c \,.\operatorname{curl} \boldsymbol{H} \,. \quad (469)$$

which of course reduces to (31a) when $q = 0$.

If we recollect that by (40) the quantity div $\boldsymbol{D}$ is equal to 4π times the space charge-density k, it is clear that its product with q represents the convection current which, as we have already seen in § 81, also exerts magnetic actions. The term involving curl $[\boldsymbol{D}, q]$ is called the "Röntgen current" as a reference to the first successful demonstration of the magnetic actions of moving polarized dielectrics.

§ 99. As we can see directly from the form of the integral law (457) and (463), according to the theory of Maxwell and Hertz the relationships between the mechanical and electromagnetic quantities are in no wise dependent on the choice of any particular co-ordinate system, or in other words the fundamental equations for the electrodynamic phenomena that occur in moving bodies are invariant when passing to a co-ordinate system moving in any way whatsoever. This peculiarity can, of course, be demonstrated in the individual differential equations (460) and (469).

As an example we shall sketch the calculation for the transformation to a co-ordinate system whose axes are parallel to the original axes and whose origin moves in the direction of the x-axis with any arbitrary variable velocity $f(t)$. We then have:

$$\left.\begin{array}{l} x = x' + f(t), \quad y = y', \quad z = z' \\ q_x = q'_x + \dot{f}(t), \ q_y = q'_y, \ q_z = q'_z \end{array}\right\} \quad . \quad . \quad (470)$$

Now the first of the equations (469) runs:

$$\dot{\boldsymbol{D}}_x + \frac{\partial(\boldsymbol{D}_x \boldsymbol{q}_y - \boldsymbol{D}_y \boldsymbol{q}_x)}{\partial y} - \frac{\partial(\boldsymbol{D}_z \boldsymbol{q}_x - \boldsymbol{D}_x \boldsymbol{q}_z)}{\partial z}$$

$$+ \operatorname{div} \boldsymbol{D} \,.\, \boldsymbol{q}_x + 4\pi \boldsymbol{J}_x = c \,.\, \left(\frac{\partial \boldsymbol{H}_z}{\partial y} - \frac{\partial \boldsymbol{H}_y}{\partial z}\right) \quad . \quad (471)$$

Since only the terms in $\dot{\boldsymbol{D}}_x$ and $\boldsymbol{q}_x$ become changed in the assumed transformation, it is sufficient to consider them exclusively. Taken together, these terms are:

$$\dot{\boldsymbol{D}}_x + \boldsymbol{q}_x \cdot \frac{\partial \boldsymbol{D}_x}{\partial x} - \boldsymbol{D}_y \cdot \frac{\partial \boldsymbol{q}_x}{\partial y} - \boldsymbol{D}_z \cdot \frac{\partial \boldsymbol{q}_x}{\partial z} \quad . \quad . \quad (472)$$

Now, by (470):

$$\dot{\boldsymbol{D}}_x = \left(\frac{\partial \boldsymbol{D}_x}{\partial t}\right)_x = \left(\frac{\partial \boldsymbol{D}_x}{\partial t}\right)_{x'} + \left(\frac{\partial \boldsymbol{D}_x}{\partial x'}\right)_t \left(\frac{\partial x'}{\partial t}\right)_x = \dot{\boldsymbol{D}}_x' - \frac{\partial \boldsymbol{D}_x}{\partial x'} \cdot \dot{f}(t).$$

If we substitute this value as well as that for $\boldsymbol{q}_x$ from (470) in (472) and observe that the terms involving the space differential coefficients of $\boldsymbol{D}$ and $\boldsymbol{q}$ remain invariant during the transformation, we get the expression:

$$\dot{\boldsymbol{D}}_{x'} + \boldsymbol{q'}_x \cdot \frac{\partial \boldsymbol{D}_{x'}}{\partial x'} - \boldsymbol{D}_{y'} \frac{\partial \boldsymbol{q'}_x}{\partial y'} - \boldsymbol{D}_{z'} \frac{\partial \boldsymbol{q'}_x}{\partial z'}$$

which proves the invariance of (472) and hence also of (471). The invariance of the other field-equations can be proved in a similar way.

The physical significance of the invariant property just discussed clearly consists in the fact that the electrodynamic processes that occur in any arbitrary stationary body or in any system of bodies moving in any way undergo no change when a common motion for the whole system of bodies becomes super-imposed on the other motions present. Hence it is not possible to detect an absolute motion of the whole system by means of electrodynamical measurements. This electrodynamic *principle of relativity* goes much further than the mechanical principle of relativity of Galilei and Newton (I, § 50). For the latter demands invariance only for uniform motions, whereas the invariance asserted here applies also

to any accelerated motions and hence also to rotations, in so far as they leave the relative motions within the system unchanged. The fact that the validity of the principle of relativity in nature has been confirmed in a great number of cases by the most refined methods of measurement that have hitherto been devised, gives strong support to the electrodynamics of Maxwell and Hertz.

§ 100. On the other hand there are certain facts which leave us in no doubt that the theory of electrodynamics here given, which was initiated by Hertz and brought to its logical conclusion by Maxwell, is untenable for bodies moving quite arbitrarily. To prove this it is fully sufficient to discuss the most striking case of all—and one which is in itself decisive—namely the propagation of electromagnetic waves in a moving rarefied gas. According to the theory of Maxwell and Hertz the electromagnetic waves should, on account of the principle of relativity, be convected along completely by the gas, since for an observer moving with the gas the waves propagate themselves, according to the principle of relativity and in agreement with all experiments, in just the same way as for an observer at rest in the gas at rest. This law should remain valid no matter how highly rarefied the gas may be, which would correspond fully with the circumstance that no material properties of the moving bodies occur in the fundamental equations of the preceding section, and that field-intensity and induction are to be practically identifiable for gases that are sufficiently rarefied. Even in the state of highest rarefaction the gas or the convected ether, respectively, remains, according to this, the carrier of the electromagnetic waves, just as of sound waves.

This assertion, however, is strongly contradicted by reality, as has been shown in particular by Fizeau in his experiments with flowing gases. There can be no question of a complete convection of electromagnetic waves by a moving gas. Rather, the propagation of waves in highly rarefied gases takes place essentially independently of

any motion of the gas. A weak influence makes itself remarked only when the gas is highly compressed. Hence if we wish to assign a particular substance as a carrier of the electromagnetic waves at all, this carrier at any rate behaves quite passively and independently as regards the motions of rarefied gases.

This circumstance can be expressed mathematically by saying that in the differential equation (460) the term involving q and the corresponding term in the differential equation (469) do not exist in reality for rarefied gases.

The recognition of this fact, which, incidentally, is nothing other than a more rigorous formulation of the theorem already indicated in § 7 that the last vestiges of gas in an evacuated space no longer exert an appreciable influence on the electromagnetic field, proved that the electrodynamics of Maxwell and Hertz was inadequate for moving bodies and that it required to be fundamentally modified.

§ 101. The solution of the difficulty that had been encountered was given by H. A. Lorentz when he set up his *electromagnetic theory of the stationary ether*. The significant idea of this theory consists in the fact that it breaks radically with the assumption that space is continuously occupied by matter, which was an essential feature of the electrodynamics of Maxwell and Hertz; this assumption was equivalent to the complete convection of the carrier of electromagnetic effects by matter. According to Lorentz's electrodynamics there is only one medium which fills the whole of space continuously and without leaving gaps and hence also penetrates through all material bodies, namely the ether. It is the ether alone which transmits all electrodynamical effects. Its state of motion is invariable. Hence it may be regarded as at rest. Matter, on the other hand, consists of separate atoms which are disposed at greater or lesser distances from one another and which move as rigid or also as deformable bodies with the velocity q with respect to the ether. They influence the electromagnetic field only in a

secondary manner, and, in fact, the interactions between the electromagnetic field and matter come about solely because matter contains and carries along with itself electric charges having the space density k. This does not exclude the possibility that such electric charges, when restricted to very small regions of space, separated from one another, may also occur outside of matter and move of themselves through the ether. The free charges are now called electrons to distinguish them from material ions. Accordingly the conception of the conduction current becomes eliminated altogether from Lorentz's theory. So electric conduction is essentially no more than electric convection; the carriers of the charge in electrolytes are the ions, and in metals they are the electrons.

Since the pure ether is identical with a pure vacuum, both the dielectric constant in it and also the magnetic permeability are equal to 1, the field-strength and induction become identical, and in place of the Maxwell-Hertz field-equations (460) and (469) we have the Lorentz equations:

$$\dot{\boldsymbol{H}} = -c \cdot \operatorname{curl} \boldsymbol{E}, \qquad \operatorname{div} \boldsymbol{H} = 0 \quad . \quad . \quad (473a)$$

$$\dot{\boldsymbol{E}} + 4\pi k \cdot \boldsymbol{q} = c \cdot \operatorname{curl} \boldsymbol{H}, \quad \operatorname{div} \boldsymbol{E} = 4\pi k \quad . \quad (473b)$$

whose great simplicity is connected in a remarkable way with their no less remarkable power. In Lorentz's theory the dielectric constant ϵ and the magnetic permeability μ lose the primary importance attaching to them in the Maxwell-Hertz theory. Rather, all the dielectric and magnetic properties of material bodies appear expressed in terms of the position and motions of the ions and the electrons in the bodies, namely in terms of electric polarization (§ 26) and the Ampérian molecular currents (§ 63). Hence the quantities ϵ and μ are not to be regarded as simple constants but rather as abbreviated terms for very complex composite expressions which emerge as certain statistical mean values arising from the combined action of an enormous number of small forces—a view which also explains physically the actual variable nature of these

quantities. This circumstance does not, of course, hinder us from using the unmodified Maxwell equations in all cases where ϵ and μ are practically constant.

In all these effects the following equation, which is obtained by combining (214) and (352), holds according to Lorentz's theory for the mechanical force exerted by an electromagnetic field with field-strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ on a point-charge of electricity e in it moving with the velocity $\boldsymbol{q}$:

$$\boldsymbol{F} = e \, . \, \boldsymbol{E} + \frac{e}{c} \cdot [\boldsymbol{q}, \boldsymbol{H}] \, . \qquad (474)$$

From this equation we can derive all the ponderomotive actions between conductors and dielectrics, magnets and currents. The results are found to agree satisfactorily with those of experiment.

§ 102. Although Lorentz's theory of the stationary ether is brilliantly confirmed in electrodynamics, a difficulty of fundamental importance arises when we enquire into the transformation of Lorentz's equations for a co-ordinate system which is moving with respect to the ether. For if we omit the terms curl $[\boldsymbol{B}, \boldsymbol{q}]$ and curl $[\boldsymbol{D}, \boldsymbol{q}]$ from the Maxwell-Hertz equations the very property of these equations becomes lost which, as we have seen, for example, in (472), provides for their invariance with respect to a motion of the co-ordinate system and hence is responsible for the validity of the principle of relativity. It is in fact immediately evident that for an observer who is moving in the same direction through the ether as a train of electromagnetic waves the velocity of propagation of the waves in the ether must be less than for a stationary observer, and we should therefore be led to expect that it should be possible to establish the velocity of the motion of the observer by electromagnetic or optical means. But even the most sensitive measurements, such as those carried out in the experiments of Michelson and Morley, failed to disclose any influence of the velocity of the earth's motion on the relative velocity of propagation of light. So it appeared for a time as if Lorentz's equations

(473) and (474) were incompatible with the principle of relativity and as if the theory would be compelled to give up one of these two most valuable foundations of its structure in order to save at least the other.

Einstein (*Ann. J. Physik*, p. 891, 1905), however, showed how it was possible to escape from this perplexing dilemma by asserting that Lorentz's equations of electrodynamics are just as completely invariant as the exact fundamental equations of mechanics for transformations from a stationary co-ordinate system to one moving with uniform velocity with respect to it, but that we may not use Galilei's transformation (I (194)) for this purpose. Instead, we must use another transformation, the Lorentz transformation, which may be regarded as a generalization of the former transformation since it becomes identical with it when $c \longrightarrow \infty$. This theorem not only effects agreement between Lorentz's electrodynamics and the principle of relativity, but also leads to a great number of other consequences, particularly for mechanics, which in fact stretch far into the realm of the theory of knowledge and which in no case whatsoever lead to a contradiction with the results of experiment, and have been many times confirmed, often in a striking manner.

It is not possible for us to go further into Einstein's principle of relativity here, as the purpose of this book is to serve as an introduction to the theory of electricity.

COMPARISON TABLE OF THE NUMERICAL VALUES OF VARIOUS QUANTITIES REFERRED TO DIFFERENT SYSTEMS OF MEASUREMENT

The critical velocity c and also all the mechanical quantities and those of energy (ponderomotive force, momentum, work, energy-density, energy-flux, heat), have the same numerical values in all the systems here represented except for a factor of a power of ten in the practical system.

Quantity.	Numerical Value in Gauss's System § 7.	Dimensions in Gauss's System.	Numerical Value in Lorentz's System of Rational Units § 7.	Numerical Value in Maxwell's Electromagnetic System § 7.	Numerical Value in the "Practical" System § 61.
Electric Intensity of Field, § 2.	E	$[C^{-\frac{1}{2}}G^{\frac{1}{2}}S^{-1}]$	$\frac{E}{\sqrt{4\pi}}$	cE	$cE \,.\, 10^{-8}$
Magnetic Intensity of Field, § 3.	H	$[C^{-\frac{1}{2}}G^{\frac{1}{2}}S^{-1}]$	$\frac{H}{\sqrt{4\pi}}$	H	H gausses
Dielectric Constant, § 2.	ϵ	$[1]$	ϵ	$\frac{\epsilon}{c^2}$	$\frac{\epsilon}{c^2} \cdot 10^9$
Magnetic Permeability, § 3.	μ	$[1]$	μ	μ	μ
Quantity of Electricity, § 11.	e	$[C^{\frac{3}{2}}G^{\frac{1}{2}}S^{-1}]$	$\sqrt{4\pi} \,.\, e$	$\frac{e}{c}$	$\frac{e}{c} \cdot 10$ ampère-secs. (Coulomb)
Space Density of Charge, § 11.	k	$[C^{\frac{1}{2}}G^{\frac{1}{2}}S^{-1}]$	$\sqrt{4\pi} \,.\, k$	$\frac{k}{c}$	$\frac{k}{c} \cdot 10$
Surface Density of Charge, § 11.	h	$[C^{-\frac{1}{2}}G^{\frac{1}{2}}S^{-1}]$	$\sqrt{4\pi} \,.\, h$	$\frac{h}{c}$	$\frac{h}{c} \cdot 10$
Electric Conductivity, § 11.	κ	$[S^{-1}]$	$4\pi\kappa$	$\frac{\kappa}{c^2}$	$\frac{\kappa}{c^2} \cdot 10^9$
Electric Induction, § 11.	D	$[C^{-\frac{1}{2}}G^{\frac{1}{2}}S^{-1}]$	$\frac{D}{\sqrt{4\pi}}$	$\frac{D}{c}$	$\frac{D}{c} \cdot 10$
Magnetic Induction, § 11.	B	$[C^{-\frac{1}{2}}G^{\frac{1}{2}}S^{-1}]$	$\frac{B}{\sqrt{4\pi}}$	B	B
Electric Potential (electromotive force) § 27.	E	$[C^{\frac{1}{2}}G^{\frac{1}{2}}S^{-1}]$	$\frac{E}{\sqrt{4\pi}}$	cE	$cE \,.\, 10^{-8}$ volt
Electric Intensity of Current, § 40.	J	$[C^{\frac{3}{2}}G^{\frac{1}{2}}S^{-2}]$	$\sqrt{4\pi} \,.\, J$	$\frac{J}{c}$	$\frac{J}{c} \cdot 10$ ampère
Electric Resistance, § 40.	w	$[C^{-1}S]$	$\frac{w}{4\pi}$	$c^2 w$	$c^2 w \,.\, 10^{-9}$ ohm.
Electric Capacity, § 16.	C	$[C]$	$4\pi C$	$\frac{C}{c^2}$	$\frac{C}{c^2} \cdot 10^9$ farads
Inductance, § 76.	L	$[C]$	$\frac{L}{4\pi}$	L	$L \,.\, 10^{-9}$ henry
Work of a Current, § 65.	A	$[C^2GS^{-2}]$	A	A	$A \,.\, 10^{-7}$ watt-seconds (joule)
Power of a Current, § 65.	A	$[C^2GS^{-3}]$	A	A	$A \,.\, 10^{-7}$ volt-ampère (watt)

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